1. Vectors

Introduction to Vectors

A vector is a mathematical object with both a **magnitude** and **direction**. Two vectors are identical as long as these two properties are the same, regardless of their start and end points.

The components of vectors can either be written as

$$\langle a, b, c \rangle$$

or as a scalar multiple of unit vectors (i,j,k)

$$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$$

Vector Operations

Determining the Norm (Magnitude) of a Vector

1. Pythagorean Theorem/Distance Formula

Finding a Unit Vector in the Same Direction as Another Vector

1. Find magnitude of vector
2. Divide that vector by its magnitude for a unit vector in the same direction as the original vector

Vector Addition

1. Add components in the same direction
    - visualized via tip-tail method or as a parallelogram

Direction Angles and Orthogonal Projections

Direction angles are the angles a vector forms with the x/y/z axes. They link the directional component of a vector with its magnitude.

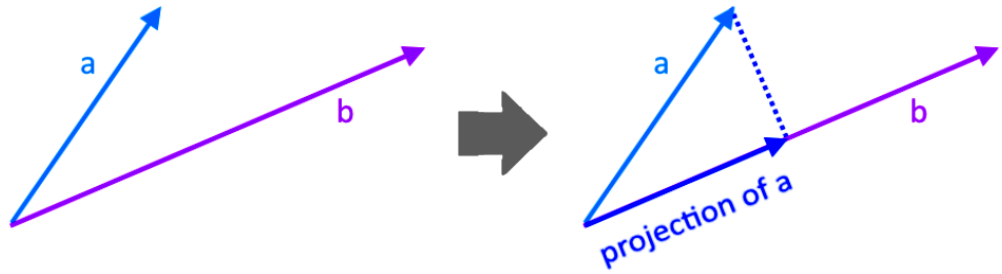
$$\cos(\alpha) = \frac{v_x}{||v||}$$

$$\cos(\beta) = \frac{v_y}{||v||}$$

$$\cos(\gamma) = \frac{v_z}{\|v\|}$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$

Orthogonal projections represent how much two vectors align.



A vector is mapped onto a subspace so that the line between the original vector and its projection is perpendicular (orthogonal) to the subspace.

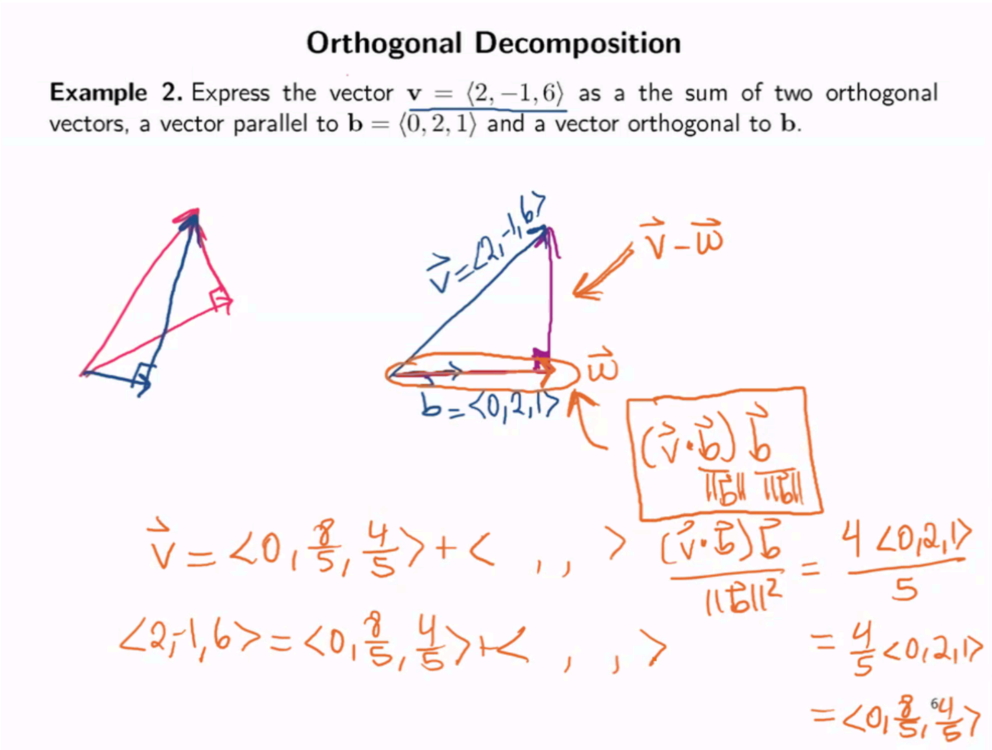
The projection vector of a onto b is

$$\text{Proj}_{\vec{b}}\vec{a} = \left(\vec{a} \cdot \frac{\vec{b}}{||b||}\right) \frac{\vec{b}}{||b||}$$

- dot product gives component of a in direction of b; division makes both b vectors unit vectors
- derivation

Orthogonal Decomposition

To break down any vector into two perpendicular components, use orthogonal projections.



]

Dot and Cross Product

Dot Product (Scalar)

Definition The dot product is an algebraic operation on two vectors, returning a scalar quantity representing a measure of how much the two vectors align with one another.

$$\mathbf{a} \cdot \mathbf{b}$$

Calculating the Dot Product The dot product can either be calculated with vector components or geometrically via magnitude + angle.

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$$

Geometric Significance

- If the dot product of two vectors is zero, the vectors are perpendicular to one another
- Cosine angle between vectors can be found by dividing dot product by magnitudes

Properties

- Commutative
- Distributive
- Scalar Multiplication

Cross Product (Vector)

Definition Binary operation on two vectors in 3D space, resulting in a vector perpendicular to both original vectors.

$$\mathbf{a} \times \mathbf{b}$$

Calculating the Cross Product Using vector components, the cross product is the determinant of a 3×3 matrix with unit vectors i/j/k in the first row

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - b_y a_z)\mathbf{i} - (a_x b_z - b_x a_z)\mathbf{j} + (a_x b_y - b_x a_y)\mathbf{k}$$

Geometric Significance Magnitude of cross product is, representing the area of the parallelogram formed by the two vectors

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$$

Properties:

- not commutative
- distributive over vector addition, scalar multiplication
- direction of resulting vector from cross product follows right-hand rule

Scalar Triple Product

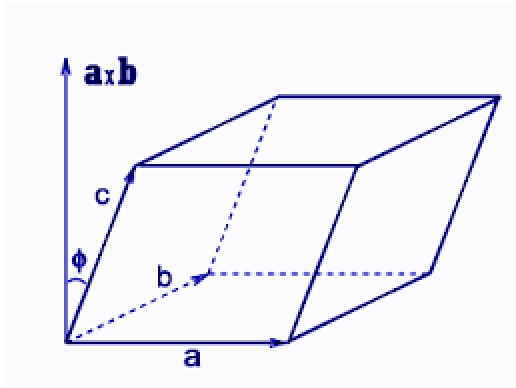
Definition A scalar triple product is

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

resulting in a scalar (dot product of **a** and (**b** x **c**)).

Calculating the Scalar Triple Product The scalar triple product is the determinant of the 3×3 matrix formed by the component of the vectors:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = a_x(b_y c_z - c_y b_z) - a_y(b_x c_z - c_x b_z) + a_z(b_x c_y - c_x b_y)$$

Geometric Significance The absolute value of the scalar triple product is the volume of the parallelepiped formed by the 3 vectors. 

A volume of 0 occurs if:

- vectors are coplanar
- any two vectors are parallel
- any one of the vectors is a zero vector

Properties

1. Cyclic permutation

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

2. Swapping order of two crossed vectors changes the sign
3. Distributive propety for vector addition (within cross product)

Specific Problem Solving

1. Equation of Plane Given Point + Normal Vector to Plane

1. Find normal vector **n** = (A, B, C) via cross product of 2 vectors on plane
2. Take a point P(x_0, y_0, z_0) on the plane
3. We know that the distance vector of point P(x_0, y_0, z_0) to any point Q (x, y, z) on the plane is always perpendicular to the normal vector
4. Using that and the dot product formula, expanding, we get

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

3. Finding Perpendicular Distance to a Plane

1. Take 2 vectors lying on the plane and take their cross product (results in a vector perpendicular to the plane)
2. Find the normal unit vector (divide normal vector by its magnitude)

3. Dot product of vector from outside point to point on the plane with normal unit vector gives perpendicular distance

4. Finding Distance from Point to a Line

1. Find vector of 2 points on the line as well as vector of outside point to a point on the line
2. Cross product for area of parallelogram
3. Divide area of parallelogram by known length of side of parallelogram (magnitude of vector of 2 points on line) ⟶ get the height of parallelogram, which is perpendicular (shortest) distance to line

Parametric and Vector Equations of Lines

For a line L containing (x, y, z) in 3-space passing through (x_0, y_0, z_0) with parallel **v** = $\langle a, b, c\rangle$ **Parametric Equation**

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

- x/y/z components of line = initial point + multiplier times component of parallel direction vector
- Derivation

From this we get the **Vector Equation**

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

Skew Lines

In 3D space, it's possible for two lines to neither be parallel nor intersect at all, forming **skew lines**. To show that two lines are skew, you must prove that they are 1) not parallel and 2) don't intersect.

Problem Solving Steps:

1. Determine whether direction vectors are scalar multiples of one another (components are proportional). If so, they are parallel.
2. Set the 2 sets of parametric equations equal to one another (equation for $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$). Determine whether the set of equations are true by solving for t_1 and t_2. If true, find the intersection by plugging t_1 or t_2 in either of the original parametric equations.

- Use the first two systems to find t_1 and t_2. Plug into the third equation in the system to ensure that the equation comes out to be true.

Prove 2 Sets of Parametric Equations Represent the Same Line

There are infinitely many ways to represent a line with parametric equations, whether by taking any point on the line or taking any parallel direction vector in the same direction.

To show that 2 sets of parametric equations are equal, you must show that they are parallel and share a point, since it's impossible for parallel lines to intersect otherwise

Problem Solving Steps:

1. Show that the two sets are parametric equations have parallel direction vectors in the same direction.
2. Show that 1 of the 2 points given are on both lines

2. Planes and Surfaces

Introduction to Planes

A plane is generally defined by a **vector normal to the plane** and a **point on the plane**.

There are two common forms of a plane, the **point-normal form** $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ and the **standard form** $Ax + By + Cz = D$.

- A, B, and C represent components of the **normal vector** to the plane
- $x_0, \ y_0, \ z_0$ represent coordinates of **point on plane**

Intersection of a Line and a Plane

To find where a line intersects a plane, substitute parametric equations of line (x/y/z = a + bt) into the plane equation (for x/y/z) and solve for the parameter t.

If t = all real values, the line lies one the plane.

Intersection of 2 Planes

If the normal vectors of the two planes are scalar multiples of each other, then the planes are parallel and there is no intersection.

To find the line formed by the intersection of 2 planes: Parametric:

1. Take the cross product of the normal directional vectors of both planes. This gives the directional vector of the line of intersection.
2. Find an intersection point of two planes and plug in for the parametric equations of a line (directional vector as coefficient of t, point as "intercept").
   - To do so, set one variable to a constant. Then, the two plane equations form a linear system of equations.

Alternatively:

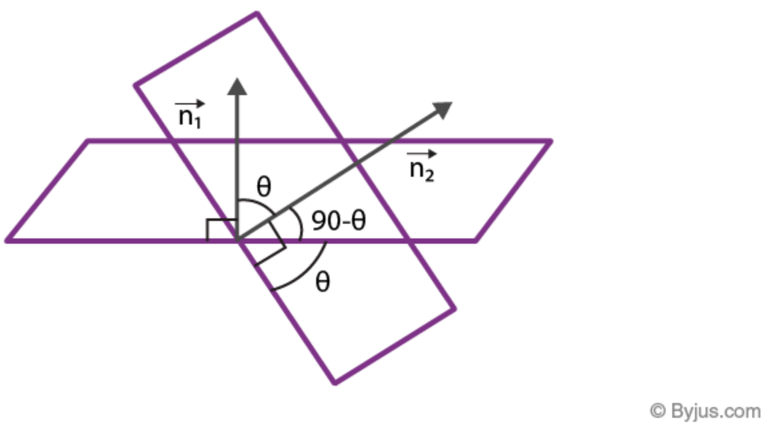
1. Take the system of equations (of the planes) and eliminate one variable. Now we're left with 2 variables, 1 equation (leave x and y).
2. Substitute equation of line into one of the original plane equations to get relationship for eliminated variable.
3. Obtain the parametric equations for line since you have relationships between all 3 variables.

To Find Direction Vector of Intersection Line: Take the cross product of the normal vectors of the two planes. Reasoning: The normal vector must be orthogonal to every vector on that plane. Therefore, the line orthogonal to both planes' normal vectors must lie on both planes. This line must be the intersection line, as it's the only line that lies on both planes.

Angle Between 2 Planes

If two planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, the acute angle between them can be found by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$



- The angle between planes is equal to the angle between their normal vectors

Distance

Between a Point and a Plane: The distance d between the point $P(x_0, y_0, z_0)$ and the plane with equation $Ax + By + Cz + D = 0$.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

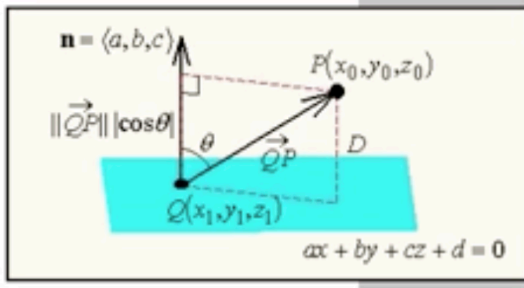
Planes: Intersection Angles and Distances

Theorem. The distance D between the point $P(x_0,y_0,z_0)$ and the plane with equation $ax+by+cz+d=0$ is given by the formula

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Proof.

$$D=\|\overrightarrow{QP}\|\,|\cos\theta|=\frac{\|\mathbf{n}\|\,\|\overrightarrow{QP}\|\,|\cos\theta|}{\|\mathbf{n}\|}=\frac{|\mathbf{n}\cdot\overrightarrow{QP}|}{\|\mathbf{n}\|}.$$



$\overrightarrow{QP}=\langle x_0-x_1,y_0-y_1,z_0-z_1\rangle$, so

$$\begin{aligned}|\mathbf{n}\cdot\overrightarrow{QP}|&=|\langle a,b,c\rangle\cdot\langle x_0-x_1,y_0-y_1,z_0-z_1\rangle|\\&=|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|\\&=|ax_0+by_0+cz_0-(ax_1+by_1+cz_1)|=|ax_0+by_0+cz_0+d|.\end{aligned}$$

Since $Q(x_1,y_1,z_1)$ lies in the plane, $ax_1+by_1+cz_1=-d$.

Since $\|\mathbf{n}\|=\sqrt{a^2+b^2+c^2}$,

$$D=\|\overrightarrow{QP}\|\,|\cos\theta|=\frac{\|\mathbf{n}\|\,\|\overrightarrow{QP}\|\,|\cos\theta|}{\|\mathbf{n}\|}=\frac{|\mathbf{n}\cdot\overrightarrow{QP}|}{\|\mathbf{n}\|}.$$

For a point Q on the plane, the distance d is equal to QP dotted with the unit normal vector.

- Denominator is norm of normal vector
- Numerator is dot product

Between 2 Parallel Planes:

Theorem. The distance D between the point $P(x_0,y_0,z_0)$ and the plane with equation $ax+by+cz+d=0$ is given by the formula

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Corollary. The distance D between planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

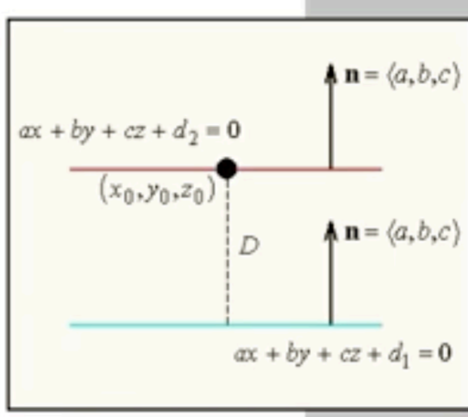
$$D=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}.$$

Proof. Let (x_0,y_0,z_0) be a point on the plane given by $ax+by+cz+d_2=0$.

According to the theorem,

$$D=\frac{|ax_0+by_0+cz_0+d_1|}{\sqrt{a^2+b^2+c^2}}$$

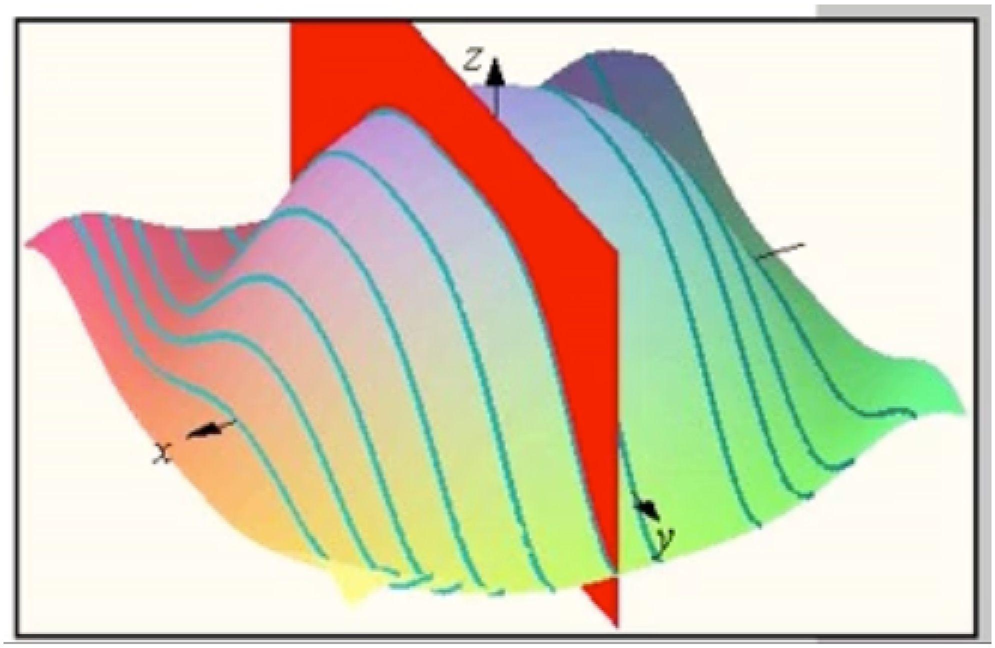
$ax_0+by_0+cz_0=-d_2$, because (x_0,y_0,z_0) lies in the plane with equation $ax+by+cz+d_2=0$.



Graphing Surfaces Using Traces

In 2D, the general shape of the curve can be obtained through having a bunch of points that lie on the curve. Similarly, in 3D, the shape of a surface can be obtained by having a bunch of curves the lie on the surface.

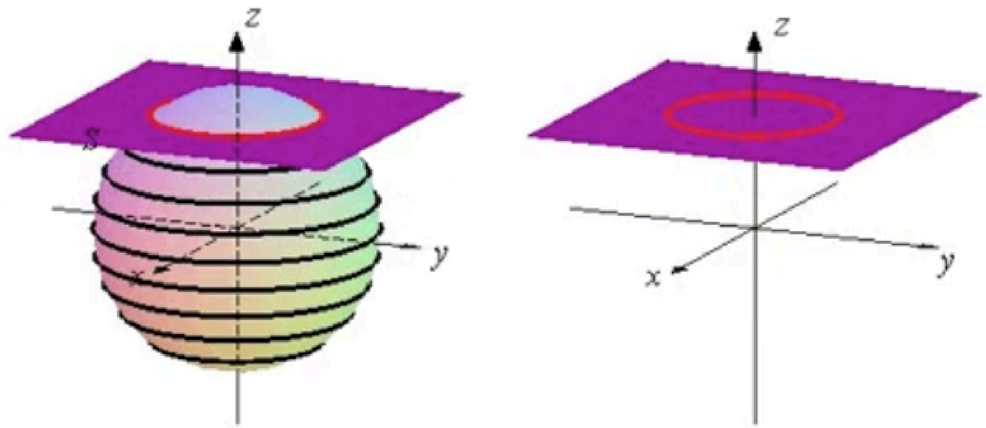
Each curve can be thought of as the intersection of a plane with the surface. The set of points on this intersection/curve is called the **trace** of the surface in that plane.



Geometric Interpretation

The goal is to take many **slices** of the surface, to the point that the **traces** provide an outline of the 3D surface.

Trace in the plane $z = k$



Planes and corresponding traces

Algebraic Interpretation

Procedure for finding the trace of a surface in a plane:

1. Determine the equation of the plane.
2. Substitute plane conditions into the equation of the surface.
3. Graph the resulting equation involving only 2 dimensions (**trace equation**)

Example 1'. Give a rough sketch of the surface S determined by

$$x^2 + y^2 + z^2 = 25.$$

Consider the plane $z = 3$: $\{(x,y,z) \mid z = 3\}$.

(x,y,z) lies in the intersection of the plane and the surface S if and only if

$$z = 3 \text{ and } x^2 + y^2 = 16.$$

The trace of our surface in the plane $z = 3$ is

$$\{(x,y,3) \mid x^2 + y^2 = 16\}.$$

Graphing Quadric Surfaces Using Traces

A general second degree equation in x, y, and z has the **general form**

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Translations for Quadric Surfaces operate the same way as 2D surfaces.

- If S is the curve for E(x,y,z), then the curve S' for E(x-a, y-b, z-c) is S translated a/b/c units in the x/y/z directions respectively.

Common Types of Quadric Surfaces

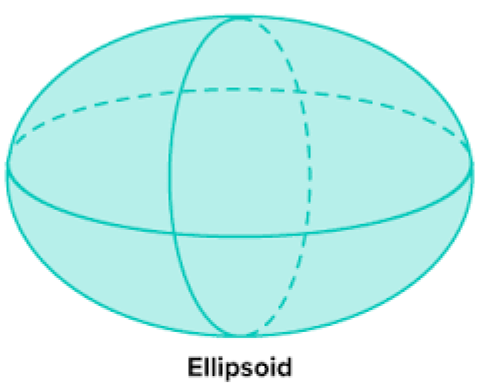
Table 11.7.2

IDENTIFYING A QUADRIC SURFACE FROM THE FORM OF ITS EQUATION

EQUATION	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
CHARACTERISTIC	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
CLASSIFICATION	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

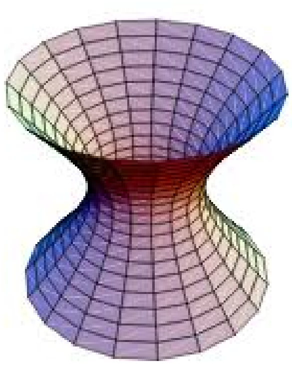
Start with sphere with fractional denominators

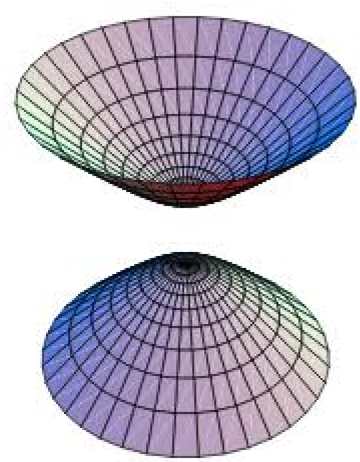
1. one minus sign (negative z) ⟶ hyperboloid of 1 sheet
2. two minus signs (z first) ⟶ hyperboloid of 2 sheets
3. remove "1" ⟶ elliptic cone
4. remove z^2 ⟶ elliptic paraboloid (parabola holds term to first degree)
5. - x^2 instead of minus ⟶ hyperbolic paraboloid (first x term changes sign)



Ellipsoid:

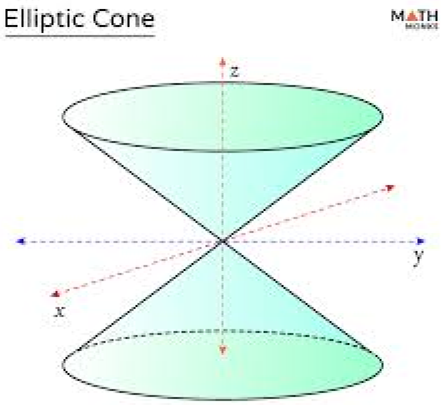
Hyperboloid of One Sheet:





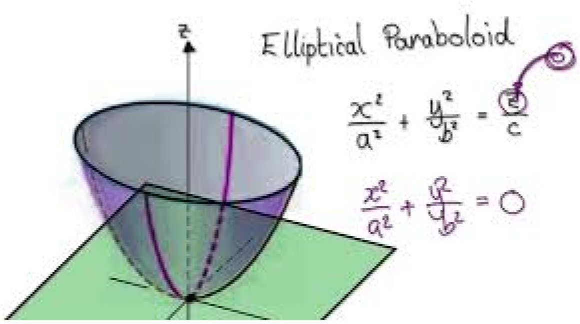
Hyperboloid of Two Sheets:

Elliptic

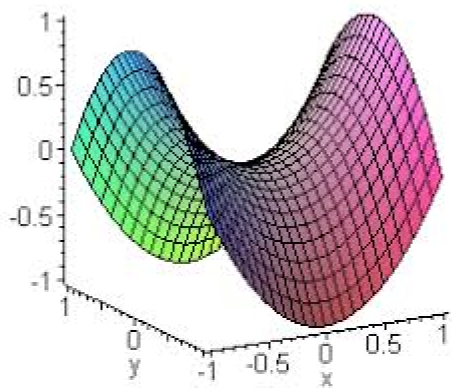


Cone:

Elliptic Paraboloid:



- horizontal cross-sections are ellipses, vertical cross-sections are parabolas



Hyperbolic Paraboloid:

- horizontal cross-sections are hyperbolas, vertical cross-sections are parabolas

3. Other 3D Coordinate Systems

Cylindrical Coordinates

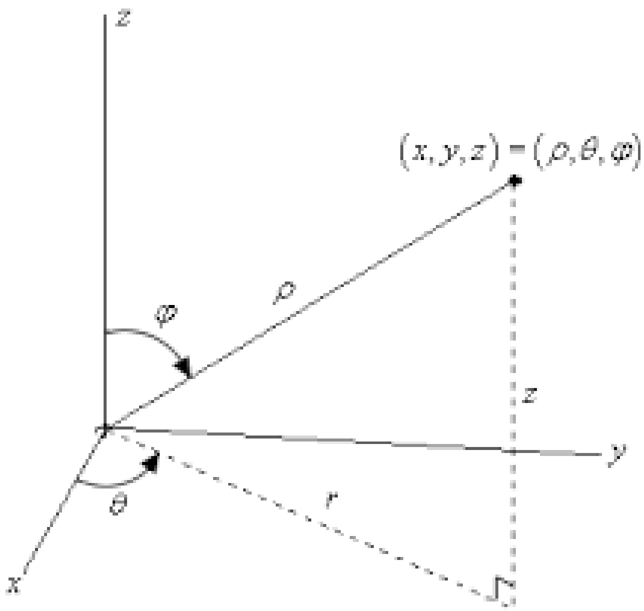
Cylindrical coordinates directly extend polar coordinates to 3-space, with coordinates consisting of (r, θ, z).

To convert from rectangular coordinates to **cylindrical coordinates**: $(r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}, z)$. To convert from cylindrical coordinates to **rectangular coordinates**: $(r\cos\theta, r\sin\theta, z)$.

Spherical Coordinates

Spherical coordinates consist of (ρ, θ, ϕ) with bounds of $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

- Theta is azimuthal angle, measured counterclockwise from positive x-axis
- Phi is polar angle, measured from positive z-axis to point



Conversion Between Cartesian and Spherical Coordinates:

- From Cartesian (x, y, z) to Spherical (ρ, θ, ϕ):
  1. $\rho = \sqrt{x^2 + y^2 + z^2}$ (the distance to the point)
  2. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (the angle in the xy-plane)
  3. $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$ (the angle from the z-axis)
- From Spherical (ρ, θ, ϕ) to Cartesian (x, y, z):
  1. $x = \rho \sin(\phi) \cos(\theta)$
  2. $y = \rho \sin(\phi) \sin(\theta)$
  3. $z = \rho \cos(\phi)$

Cylindrical to Spherical: $(r = \rho \sin \phi, \theta = \theta, \phi = \frac{\pi}{2})$

4. Vector-Valued Functions

Introduction

A **vector-valued function** is any function with scalar inputs whose outputs are vectors.

$$\mathbf{r}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$$

$$\mathbf{r}(t) = \langle f_1(t),\ f_2(t),\ f_3(t)\rangle$$

- The domain of the vector-valued function is the intersection of domains of its component functions.
- Two vector-valued functions may trace the same graph/orientation but can be different from one another (e.g., traverse same route at dif. speeds)

The output of a vector-valued function is a set of points forming a curve C with an orientation (in increasing t). In this case, r is called the **position vector** or **radius vector** for C.

Graphs

The graph of a vector-valued function is obtained by plotting the **endpoints of the vector $\mathbf{r}(t)$ starting on the origin** for every domain value of t. In this way, the components of the vectors which fulfill the function can form points.

The vector valued function may be broken into a set of parametrics for x/y/z.

$$x = f_1(t),\ y = f_2(t),\ z = f_3(t)$$

At the same time, symmetric equations are derived from parametrics ($x = x_0 + at, y = y_0 + bt, z = z_0 + ct$):

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Calculus of Vector-Valued Functions

Analogous Limits, Derivatives, and Integrals From Real-Value Functions

Limits/derivatives/integrals can essentially be "distributed" among the components of the vector-valued function:

$$\lim_{t\to a}\mathbf{r}(t) = \left(\lim_{t\to a} f_1(t)\right)\mathbf{i} + \left(\lim_{t\to a} f_2(t)\right)\mathbf{j} + \left(\lim_{t\to a} f_3(t)\right)\mathbf{k}$$

$$\mathbf{r}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}$$

$$\int \mathbf{r}(t)\,dt = \int f_1(t)\,dt\,\mathbf{i} + \int f_2(t)\,dt\,\mathbf{j} + \int f_3(t)\,dt\,\mathbf{k}$$

$$\int_a^b \mathbf{r}(t)\,dt = \int_a^b f_1(t)\,dt\,\mathbf{i} + \int_a^b f_2(t)\,dt\,\mathbf{j} + \int_a^b f_3(t)\,dt\,\mathbf{k}$$

Requirements:

- $\lim_{t \to a} \mathbf{r}(t)$ exists if and only if the limit for all its components also exists
- $\mathbf{r}$ is differentiable at some point t_0 if and only if all its components are also differentiable at t_0
- $\mathbf{r}$ is only integrable if and only if all its components are also integrable

Additionally, a vector-valued function is only continuous at point "a" if and only if all of its components are also continuous at "a".

In all instances, the derivative of vector-valued function $\mathbf{r}$ has the same differentiation rules as for a regular function f(x). The same applies for integration.

- Note: Rule also apply even if one function is scalar and the other is vector-valued

Geometry of Derivatives

If $\mathbf{r}$ is a vector-valued function, then the derivative of $\mathbf{r}$ is given by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Let *P* be a point on the graph of a vector-valued function $\mathbf{r}$, and let $\mathbf{r}(t_0)$ be the position vector from the origin to *P*. If $\mathbf{r}'(t_0)$ exists and is nonzero, then $\mathbf{r}'(t_0)$ is the **tangent vector** to the graph of $\mathbf{r}$ at $\mathbf{r}(t_0)$.

- The line through *P* parallel to tangent vector is the tangent line to the graph of $\mathbf{r}$

Additionally, $\mathbf{r}'(t)$ is orthogonal to the postion vector $\mathbf{r}(t)$ at all points **if the curve of r(t) lies on a sphere.**

Extension to Dot and Cross Product

$$(\mathbf{r}_1 \cdot \mathbf{r}_2)' = (\mathbf{r}_1' \cdot \mathbf{r}_2) + (\mathbf{r}_1 \cdot \mathbf{r}_2')$$

$$(\mathbf{r}_1 \times \mathbf{r}_2)' = (\mathbf{r}_1' \times \mathbf{r}_2) + (\mathbf{r}_1 \times \mathbf{r}_2')$$

- Order of crosses matter

Parametrization of Curves

The parametrization of vector-valued functions should be thought of as position/velocity/acceleration regarding the original function, first derivative, and second derivative. The parameter is usually **t**, time.

Whether a Vector-Valued Function is Smooth

A vector-valued function **r(t)** is smooth if:

1. The components of **r(t)** are differentiable ($\mathbf{r}'$ is a continuous function **defined** at all points in the domain of $\mathbf{r}$)
2. The derivative **r'(t)** does not equal the zero vector for any **t** in the given interval
   - If 3 components of the first derivative are zero at specific **t** value(s) then the parametrization of the curve is **not smooth**

There are many different parametrizations of curves, which all work as long as they have the same range of *C*.

Composition of Parametrizations

For $\mathbf{r}_2 = \mathbf{r}_1 \circ g$ to be a smooth parametrization of *C*

- The range of the inner function must lie within the domain of the outer function.

For chain rule to be applicable to differentiate this composed function $\mathbf{r}_2 = \mathbf{r}_1 \circ g$:

- Inner function must be differentiable at points where it is defined
- Outer function must be differentiable at range of g(t)

Parametrization of a Curve in Terms of Arc Length

The following takes the assumption of smooth curves.

Finding Arc Length of Parametric Curve If $\mathbf{r}(t)$, $a \leq t \leq b$, parametrizes the curve C, then the arc length s of C can be written

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt$$

where $|\mathbf{r}'(t)|$ is

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The upper bounds of a/b represent the start/end points of the arc parameter (to find **arc length between 2 points**).

Reparametrization by Arc Length Reparametrization describes the position based on **actual distance traveled** along the curve from a starting point.

Process of Reparametrization:

1. Find arc length function s(t)
   - Indefinite integral of $\mathbf{r}'(t)$
2. Find the inverse; get the original parameter "t" in terms of the arc length "s"
3. Express the original curve (in terms of t) now in terms of s: r(t) → r(s)

Result: Arc length reparametrization in same direction as given line with reference point

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

Reversing the Direction of Parametrization For a parametrization $r(t)$ on the interval $a \leq t \leq b$, if we want to find a reverse parametrization $g(\tau)$, we need a transformation where $t = g(\tau)$ such that the starting value of $t = a$ corresponds to $\tau = b$.

Unit Tangents, Normals, and Binormals

Unit Tangents

For two smooth parametrizations of curve *C* with the same orientation, their tangent lines will be in the same direction but possibly different lengths.

The unit tangent vector describes the direction of the curve at a given point, with a magnitude of 1.

Definition The tangent vector divided by its magnitude:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

- $\mathbf{r}(t)$ is the position vector of the curve as a function of (t).
- $\mathbf{r}'(t)$ is the derivative of the position vector, representing the velocity vector (the direction of motion of the particle along the curve).
- $\|\mathbf{r}'(t)\|$ is the magnitude of the velocity vector.

Note: $\mathbf{T}(t)$ should be found in such an order: $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{T}(t)$, and then plug in point t

Unit Normal Vector

The principal unit normal vector describes the direction in which the curve is bending at a given point. It is **perpendicular** to the tangent line and points in the direction which the 3D curve will bend in (direction of concavity).

Definition The derivative of the unit tangent vector (analogous to second derivative) divided by its magnitude:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

- $\mathbf{T}'(t)$ is the derivative of the unit tangent vector with respect to (t), representing the change in direction of the tangent vector.
- $\|\mathbf{T}'(t)\|$ is the magnitude of $\mathbf{T}'(t)$.

Note: $\mathbf{N}(t)$ is only defined when $\mathbf{T}'(t) \neq 0$. If it is zero, that would mean a constant tangent vector and a linear parametrization, which would have no curvature.

Binormal Vector

The binormal vector is perpendicular to both the tangent and normal vectors, completing the orthonormal basis for the curve.

Definition:
The binormal vector is the cross product of both the tangent vector and the normal vector, tangent to both

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$

- magnitude of 1 since both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are 1 (thus only indicates direction)
- indicates the axis of torsion

Osculating, Normal, and Rectifying Planes

Osculating Plane contains both tangent $\mathbf{T}(t)$ and normal $\mathbf{N}(t)$ (first/second unit derivatives).

Normal Plane contains both normal $\mathbf{N}(t)$ and binormal $\mathbf{B}(t)$.

Rectifying Plane contains both tangent $\mathbf{T}(t)$ and binormal $\mathbf{B}(t)$.

Their equation are constructed through a point (at specified *t* value) and the normal vector (which is whatever one is not included on the plane, since tangent/normal/binormal are all orthogonal to one another).

Curvature

Definition and Significance

Curvature is a scalar quantity that measures how sharply a curve bends at a given point. It is how rapidly the tangent vector changes moving along the arc length of the curve.

$$\kappa(t) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|$$

Note: Curvature's parameter is really arc length, not parameter *t*

Curvature of a Smooth 2D Parametric Curve

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

Curvature of a Plane Curve (Cartesian Coordinates)

$$\kappa(x) = \frac{|d^2y/dx^2|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

Curvature of 3D Curve

$$\kappa(t = t_0) = \frac{\|\mathbf{T}'(t_0)\|}{\|\mathbf{r}'(t_0)\|} = \frac{\|\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)\|}{\|\mathbf{r}'(t_0)\|^3}$$

Interpretation

- The larger the magnitude, the greater the change in tangent vector over a short segment of the curve.

Osculating Circle

An **osculating circle** is the circle that "best fits" a curve at a give point. They share the same **tangent** and **curvature**.

The curvature of a circle is always 1/r, so to find the osculating circle, use the point on the curve and 1/r = κ.

Motion Along a Curve

Velocity

To analyze motion along a curve, it is necessary to know both its position and velocity at that instant.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

- Speed is first derivative of displacement, equivalent to the product of speed with unit tangent vector

Acceleration

The acceleration of a particle lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ (tangential and centripetal acceleration).

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

The definition in terms of tangential and centripetal acceleration:

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa(t)\left(\frac{ds}{dt}\right)^2\mathbf{N}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t)$$

- $a_T(t)$ represents the tangential component of acceleration.
- $a_N(t)$ represents the normal component of acceleration.
- $\mathbf{T}(t)$ is the unit tangent vector.
- $\mathbf{N}(t)$ is the principal normal vector.

If a particle has position as a function of time t given by the smooth vector-valued function $\mathbf{r}(t)$, then at each time t, the vectors $\mathbf{v}$ and $\mathbf{a}$, and the scalars κ, a_T, and a_N are related by:

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} \Rightarrow \mathbf{a}_T = a_T\mathbf{T}$$

$$a_N = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} \Rightarrow \mathbf{a}_N = a_N\mathbf{N}$$

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

- a_T is the tangential acceleration.
- a_N is the normal acceleration.
- κ is the curvature of the particle's path.

Distance vs. Displacement

Distance Travelled

$$\text{Arc Length} = \int_a^b \|\mathbf{r}'(t)\|$$

- This is the same formula as the arc length formula in 3-space, where it is the square root of component derivatives squared
- **Integral of speed**

Magnitude of Net Displacement Vector

$$\text{Straight-Line Distance} = \left\| \int_a^b \mathbf{r}'(t) dt \right\|$$

- **Integral of velocity**
- Alternatively, find Δr using $\mathbf{r}(b)$ - $\mathbf{r}(a)$ and determine its norm

1. Multivariate Functions

Introduction

Definition

A **multivariable function** is a function with more than one input variable.

- Commonly represented as $f(x, y)$ for functions of two variables or $f(x, y, z)$ for functions of three variables.

Domain and Range

Domain: The set of all possible input values (tuples) for which the function is defined.

- For $f(x, y) = \sqrt{4 - x^2 - y^2}$, the domain is the set of points (x, y) such that $x^2 + y^2 \leq 4$.
- The maximum domain of a multivariable function with n variables is $\mathbb{R}^n$, representing a set of n-tuples of all numbers

Range: The set of possible output values for the function.

Graphs

- **Graph of $f(x, y)$**: A surface in three-dimensional space representing points (x, y, z) where $z = f(x, y)$.
  - Example: For $f(x, y) = x^2 + y^2$, the graph is a paraboloid opening upwards.

Level Curves (Contour Maps) Since it's difficult to plot some of these curves, level curves are used to give a better idea of the shape of the multivariable function in 2D.

Level Curves (or **Contour Lines**) represent points where different inputs have a constant output.

- For a function $f(x, y)$, a level curve for constant k is a set of all points (x, y) such that $f(x, y) = k$.
- Example: For $f(x, y) = x^2 + y^2$, the level curves are circles $x^2 + y^2 = k$.

Sets in 2-Space and 3-Space

Types of Points

Interior Point

- A point p is an **interior point** of a set D if there exists a neighborhood around p that is entirely contained within D.
- Formally, there is a small radius $r > 0$ such that all points within this radius are in D.

Boundary Point

- A point p is a **boundary point** of a set D if, for every neighborhood around p, there are points **both** in D and outside D.
- Intuitively, a boundary point is on the "edge" of D and lies between points inside and outside the set.

Accumulation Point (or Limit Point)

- A point p is an **accumulation point** of a set D if every neighborhood around p contains **infinitely many points** from D, no matter how small the neighborhood is.
- If D has only a finite number of elements, it has no accumulation points.

Intervals and Their Types

- **Closed Interval**: Includes all its boundary points. Denoted as $[a, b]$.
- **Open Interval**: Includes none of its boundary points. Denoted as (a, b).
- **Neither Open Nor Closed**: A set D is neither open nor closed if:
  - $D = \mathbb{R}^2$ or $D = \mathbb{R}^3$ (depending on context).
  - D is the empty set $\emptyset$.

2. Partial Derivatives

Limits of Multivariable Functions

Definition

- The **limit** $\lim_{(x,y)\to(a,b)} f(x,y)$ is the value that $f(x,y)$ approaches as (x,y) gets arbitrarily close to (a,b).
- The limit only exists if it holds for all paths approaching that specific point
  - Note: The path must actually lead to the point (a,b), and cannot simply approach from any direction

For $f(x,y) = \frac{xy}{x^2+y^2}$:

- Approaching $(0,0)$ along $y = x$: $f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$.
- Approaching $(0,0)$ along $y = -x$: $f(x,-x) = \frac{-x^2}{2x^2} = -\frac{1}{2}$.
- Since the limits differ, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Testing for the Existence of a Limit

Method 1: Converting to Polar/Spherical Coordinates Convert $f(x,y)$ into polar coordinates to rewrite the limit as $\lim_{r\to 0}$. If the limit of $f(x,y)$ exists and only depends on r, then the limit exists and is independent of direction. If the limit varies with θ, then the limit evaluates to something different from a different path.

- Typically, if the limit exists through checking with this method, it evaluates to 0

Method 2: Evaluate Along Specific Paths

1. Check along coordinate axes
2. Try different paths like $y = x$ or $y = x^2$. Substitutions that lead to an equivalent degree in the numerator and denominator are usually helpful.
   - Note: Not every path can be attempted. The path must be able to lead to the point (x,y,z) that the limit is actually aproaching.

Evaluation of Limit

If continuous, directly plug in the coordinate that the variables are approaching. If above methods for testing the existence of a limit reach a constant value for an existing limit, then that is the evaluated limit.

Finding Removable Discontinuities

Requirements:

1. The limit of the function as x/y approaches x_0/y_0 exists
2. Graph is not continuous at $f(x_0, y_0)$, whether it's because $f(x_0, y_0)$ does not equal the limit or the point doesn't exist.

Continuity of Multivariable Functions

Definition of Continuity

A function $f(x, y)$ is **continuous at a point** (a, b) if:

- $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$
  - limit exists, point exists, and the two are equivalent
  - continuity depends on whether the function approaches the same value from all directions in the domain.

Determining Continuity of a Multivariate Function

Theorem 1: If f_1 and f_2 are continuous functions, then any function formed by applying the basic operations of addition, subtraction, multiplication, or division will also be continuous due to the properties of limits.

Theorem 2: If g and h are single variable functions, g is continuous at x_0, and h is continuous at y_0, then any function formed by applying the basic operations will also be continuous.

Theorem 3: If g is a single variable function continuous at $h(x_0, y_0)$ and h is a two-variable function continuous at (x_0, y_0), then the composite function $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0).

Partial Derivatives

Partial derivatives measure rate of change for one variable while holding all others constant.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Notation

For $f(x, y)$:

- $\frac{\partial f}{\partial x}$ or f_x: Partial derivative with respect to x with all else constant
- $\frac{\partial f}{\partial y}$ or f_y: Partial derivative with respect to y with all else constant

For $f(x, y) = x^2y + y^3$:

- $\frac{\partial f}{\partial x} = 2xy$
- $\frac{\partial f}{\partial y} = x^2 + 3y^2$

Higher-Order Partial Derivatives

Subscript and Partial Notation

Subscript Notation:

- In subscript notation, the order of differentiation is **from left to right**.
- For example, in f_{xy}, the function is first differentiated with respect to x and then with y

Partial Notation:

- In partial derivative notation, the order of differentiation is **from right to left**.
- For example, in $\frac{\partial^2 f}{\partial y \partial x}$, the function is first differentiated with respect to x and then with y

For $f(x, y) = x^2y + y^3$:

- $\frac{\partial^2 f}{\partial x^2} = 2y$
- $\frac{\partial^2 f}{\partial y^2} = 6y$
- $\frac{\partial^2 f}{\partial x \partial y} = 2x$

Mixed Higher Order Derivatives

If all mixed partials are continuous in a neighborhood, then all the mixed partials are equal, regardless of the order they were taken in.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Chain Rule for 3 Dimensions

Application of Chain Rule

Let f be a function of x and y, where x and y are in terms of t. If everything is differentiable at (x_0, y_0):

$$\frac{d}{dt} f(x(t), y(t))\Big|_{t=t_0} = f_x(x_0, y_0) \cdot x'(t_0) + f_y(x_0, y_0) \cdot y'(t_0)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Usage in Implicit Differentiation

If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\frac{\partial f}{\partial y} \neq 0$, then:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

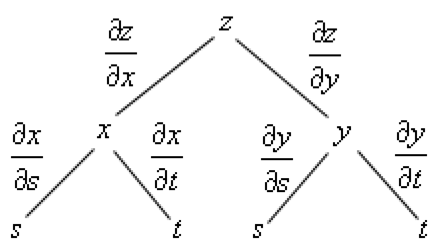
- Also very convenient for implicit differentiation for function y of one variable x

Chain Rule for Multivariate Functions

$$\frac{\partial}{\partial u} f\big(x(u,v), y(u,v)\big) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u},$$

$$\frac{\partial}{\partial v} f\big(x(u,v), y(u,v)\big) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Use the following tree to better understand components of the chain rule:



3. Tangent Planes and Local Linear Approximations

Local Linear Approximation for 3-Dimensional Space

A function $f(x, y)$ is said to be differentiable at a point (x_0, y_0) if small changes in x and y produce a change in $f(x, y)$ that can be approximated by a linear function of dx and dy.

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

Total Differential

The **total differential** df represents the best linear approximation to the change in $f(x, y)$ for small changes in x and y. Generally, you are given a point $P(x_0, y_0)$, and a point $Q(x, y)$ to approximate for using (x_0, y_0). If $dx = x - x_0$ and $dy = y - y_0$, then:

$$f(x, y) \approx f(x_0, y_0) + \Delta df$$

where,

$$\Delta df = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

- Essentially a linear combination of the partial derivatives of f, weighted by the infinitesimal changes in x and y.

Defining and Finding the Tangent Plane

Suppose (x_0, y_0, z_0) is a point on the surface S, and H is a plane containing (x_0, y_0, z_0). H is called a **tangent plane** to S at (x_0, y_0, z_0) if, for every smooth curve C lying on S and passing through (x_0, y_0, z_0), the tangent line to C at (x_0, y_0, z_0) lies in H.

Theorem. Let (x_0, y_0, z_0) be any point on the surface $z = f(x, y)$. If f is differentiable at (x_0, y_0), then the surface has a tangent plane at (x_0, y_0, z_0), and the equation of this plane is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

- All tangent lines at that point will lie in the tangent plane.
- Point (x_0, y_0, z_0) on plane, with z_0 being $f(x_0, y_0)$.
- Normal vector of tangent plane is $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$ for any surface in the form $z = f(x, y)$.

Similarly, if a surface is defined implicitly by an equation of the form $F(x, y, z) = 0$, then the tangent plane to the surface at a point (x_0, y_0, y_0) is given by the equation:

$$\frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0) = 0$$

- (x, y, z) is any other point on the plane whereas (x_0, y_0, z_0) is the point of tangency (where the plane touches the surface)

Finding Parametrics of Line Normal to Surface at Point (Intersection of Surface and Tangent Plane)

1. Use partial derivatives as the slope.
2. Use point on the plane as point.

4. The Gradient and Directional Derivatives

The Gradient of a Function

The **gradient** of a differentiable function $f(x, y, z)$ is a **vector** defined as:

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Definition and Notation of the Directional Derivative

If f is differentiable at (x_0, y_0) and $\mathbf{v}$ is a nonzero vector in 2-space, then the instantaneous rate of change of f at (x_0, y_0) along directional vector $\mathbf{v}$ is called the **directional derivative** of f at (x_0, y_0) in the direction of $\mathbf{v}$, and is denoted $D_{\mathbf{v}} f(x_0, y_0)$.

In the form $z = f(x, y)$, the directional derivative is given by:

$$D_{\mathbf{v}} f(x, y) = \nabla f(x, y) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\partial f}{\partial x} \frac{\mathbf{v}_x}{\|\mathbf{v}\|} + \frac{\partial f}{\partial y} \frac{\mathbf{v}_y}{\|\mathbf{v}\|}.$$

For a function $f(x, y, z)$ and a direction vector $\mathbf{v} = (a, b, c)$, mathematically, the directional derivative $D_{\mathbf{v}} f(x, y, z)$ of f at the point (x, y, z) in the direction of the vector $\mathbf{v}$ is:

$$D_{\mathbf{v}} f(x, y, z) = \nabla f(x, y, z) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\partial f}{\partial x} \frac{\mathbf{v}_x}{\|\mathbf{v}\|} + \frac{\partial f}{\partial y} \frac{\mathbf{v}_y}{\|\mathbf{v}\|} + \frac{\partial f}{\partial z} \frac{\mathbf{v}_z}{\|\mathbf{v}\|}$$

- Dot product of gradient of function with unit vector in direction of $\mathbf{v}$
- Note that only the direction of the direction vector matters; scaling it will not affect the directional derivative

Using Gradients to Determin the Tangent Vector to Curve of Intersection

Find the gradients (vectors normal) to both surfaces. Crossing the gradients will return a **unique direction vector** which will be tangent to the surface of intersection.

- Scalar multiples also work, but it is the direction that matters and is unique. Additionally, both $\mathbf{T}$ and $-\mathbf{T}$ work.

Geometric Significance of the Gradient

Direction of Steepest Change, Magnitude Being Steepest Slope Following from the definition of the directional derivative, we are able to see that it is **maximized** when travelling in the direction of the gradient, and that the magnitude of the gradient itself is the magnitude of the steepest slope.

At each point (x, y) where $\nabla f(x, y) \neq 0$, the gradient $\nabla f(x, y)$ gives the **direction** of the maximum increase in slope of the surface $z = f(x, y)$, and the **rate of change** is $\|\nabla f(x, y)\|$. Similarly, the steepest decrease in slope is the negative gradient $-\nabla f(x, y)$ with the same magnitude.

- It is important to realize that this is taken from a specific point (x, y), in consideration of the paths starting from that point.

Relative Extrema or a Saddle Point If $\nabla f(x, y) = 0$, then (x, y) is either a relative maximum, relative minimum, or a "saddle point".

- A saddle point is where coming from one direction, the function is a relative maximum, but from another, it is a relative minimum.

Orthogonal to Level Curves At Each Point on Curve A level curve is a set of all points with the same elevation. To have the steepest change in elevation, the gradient will point in a direction orthogonal to the level curve.

5. Extension to 3+ Variables

Differentiability and Continuity

Differentiability Suppose f is a function of n variables and its first partials exist at each point in some n-dimensional open ball centered at $(a_1, \ldots, a_n)$. If these partials are continuous at $(a_1, \ldots, a_n)$, then f is differentiable at $(a_1, \ldots, a_n)$.

Continuity If f is a function of n variables, differentiable at $(a_1, \ldots, a_n)$, then f is continuous at $(a_1, \ldots, a_n)$.

Chain Rule for Partial Derivatives

Each of the *n* Variables are Differentiable Functions of a Single Variable *t* If f is a differentiable function of the n variables $x_1, \ldots, x_n$, and each of these are differentiable functions of the variable t, then the composition $f(x_1(t), \ldots, x_n(t))$ is differentiable, and

$$\frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{dx_j}{dt}.$$

A More General Form for Any Number of *t* If f is a differentiable function of the n variables $x_1, \ldots, x_n$, and each of these are differentiable functions of the m variables $t_1, \ldots, t_m$, then the composition $f(x_1(t_1, \ldots, t_m), \ldots, x_n(t_1, \ldots, t_m))$ is differentiable, and for each i,

$$\frac{\partial}{\partial t_i} f(x_1(t_1, \ldots, t_m), \ldots, x_n(t_1, \ldots, t_m)) = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t_i}.$$

Total Differentials

If f is a function of the n variables $x_1, \ldots, x_n$, and f is differentiable at $(a_1, \ldots, a_n)$, then the total differential of f at $(a_1, \ldots, a_n)$, denoted df, is defined to be:

$$df = \frac{\partial f}{\partial x_1}(a_1, \ldots, a_n)dx_1 + \cdots + \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n)dx_n = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a_1, \ldots, a_n)dx_j.$$

- Think of df as a function of the weighted variables $dx_1, \ldots, dx_n$.

Goodness of Approximation If, for any $(x_1, \ldots, x_n)$, $\Delta f = f(x_1, \ldots, x_n) - f(a_1, \ldots, a_n)$, and for each i, $dx_i = x_i - a_i$, then (appropriately interpreted):

$$\Delta f \approx df, \quad \text{when } dx_i \approx 0 \text{ for all } i.$$

Directional Derivatives and Gradients

Gradient

$$\nabla f(a_1, a_2, \ldots, a_n) = \left\langle \frac{\partial f}{\partial x_1}(a_1, a_2, \ldots, a_n), \ldots, \frac{\partial f}{\partial x_n}(a_1, a_2, \ldots, a_n) \right\rangle$$

- Gradient is normal to the tangent plane of a level surface at point (x_0, y_0, z_0).

Directional Derivative

$$D_{\mathbf{u}} f(a_1, \ldots, a_n) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a_1, \ldots, a_n) u_j = \nabla f(a_1, \ldots, a_n) \cdot \mathbf{u}$$

- Where $\mathbf{u}$ is a unit direction vector (to reduce into unit vector, magnitude is calculated the same way).
- Maximum and minimum directional derivatives remain the magnitude of gradient in its direction.

6. 3D Extrema and Second Partials

Extrema of Functions of Two Variables

Determining Whether Extrema Exist on the Interval

Extreme Value Theorem If $f(x, y)$ is defined and continuous on a closed and bounded region R, then f has both an absolute maximum and an absolute minimum on R.

- The function may have either an interior or boundary relative extrema.

Critical Values A point (x_0, y_0) is called a **critical point** of the function f if $\nabla f(x_0, y_0) = 0$ **or** if one or both of the first partials does not exist at (x_0, y_0).

- Every relative extremum of the function f occurs either at a boundary point of $\mathrm{dom}(f)$ or at a critical point.
- Not every critical point is necessarily a relative extremum (e.g., saddle point).

If f has an interior relative extremum at (x_0, y_0), **and** if f_x and f_y both exist at (x_0, y_0), then:

$$\nabla f(x_0, y_0) = \langle 0, 0 \rangle \quad \text{(Both partials are zero.)}$$

Finding Extrema

1. Clarify the function $f(x, y)$ and determine its domain and boundaries.
2. Find all critical points by: 1) Setting partials to zero. 2) Finding when either partial is undefined. 3) Finding critical points on boundary conditions [absolute extrema only].
3. Classify each critical point using the Hessian determinant (second partials test): $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$. **This is unnecessary when determining absolute extrema.** (skip to step 4)
4. Compare function values.

Proving whether a relative max/min is absolute max/min:

1. Check asymptotic behavior.
2. Check against other critical points and boundaries.

Second Partials Test

Determines whether a given point is an extremum and whether that is a maximum, minimum, or saddle point.

Let $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$.

- **Case 1:** If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a **relative minimum** at (x_0, y_0).
  - $f_{yy}(x_0, y_0)$ must also be positive. Slope along both x and y directions is increasing.
- **Case 2:** If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a **relative maximum** at (x_0, y_0).
  - $f_{yy}(x_0, y_0)$ must also be negative. Slope along both x and y directions is decreasing.
- **Case 3:** If $D < 0$, then f has a **saddle point** at (x_0, y_0).
  - Either f_{xx} or f_{yy} is negative.
- **Case 4:** If $D = 0$, then the second partial test is **inconclusive**.

7. Lagrange Multipliers for Extrema

Concept

Lagrange multipliers are used to solve **constrained optimization** problems, where you want to find the extrema of an objective function $f(x, y, z)$ while following a constraint/condition $g(x, y, z) = 0$.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

Alternatively, define the **Lagrangian** as:

$$\mathcal{L}(x, y, \ldots, \lambda) = f(x, y, \ldots) - \lambda g(x, y, \ldots)$$

To find the critical points, we take the partial derivatives of the Lagrangian and set them to zero for a system of equations.

$$\frac{\partial \mathcal{L}}{\partial x} = f_x - \lambda g_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = f_y - \lambda g_y = 0$$

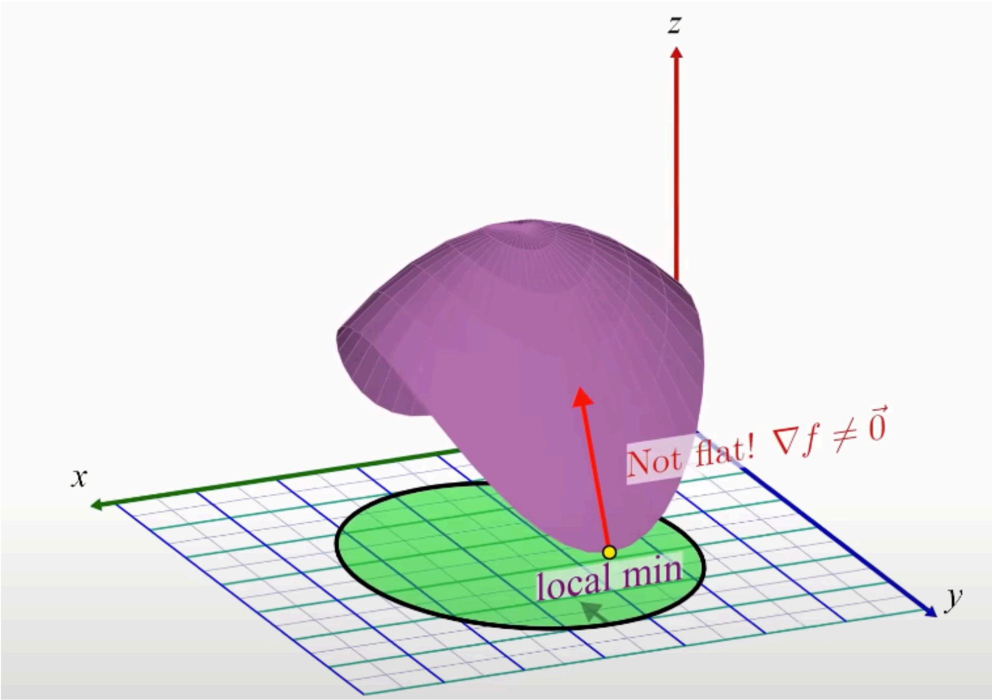
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -g(x, y, \ldots) = 0$$

- This last equation is the constraint itself.

Reasoning

Lagrange multipliers are used to avoid excessive substitution in constrained optimization.

The idea is that for any function $f(x, y)$ in the real world, there will often be restrictions on x and y for where it may go.



- The green surface (constraint) is a level curve.

The level curve $f(x,y)=c$ must be tangent to $g(x,y)$, because otherwise moving in one direction along $g(x,y)$ will increase or decrease the objective function. Therefore, all gradients of the constraint function will be parallel to the gradients at the extrema of the objective function.

Problem-Solving Procedure

1. Determine both the **objective function** $f(x,y)$ and the **constraint function** $g(x,y)=0$.
2. Set $\mathcal{L}(x,y,\ldots,\lambda)=f(x,y,\ldots)-\lambda g(x,y,\ldots)=0$.
3. Solve the system by isolating λ in terms of other variables. Solve the system of equations.
4. Determine the critical points (x,y,z). Evaluate to determine whether they are local maximums/minimums.
    - To classify, either compare with other evaluated function values or use determinant of Hessian matrix.

1. Double Integrals (Iterated Integrals)

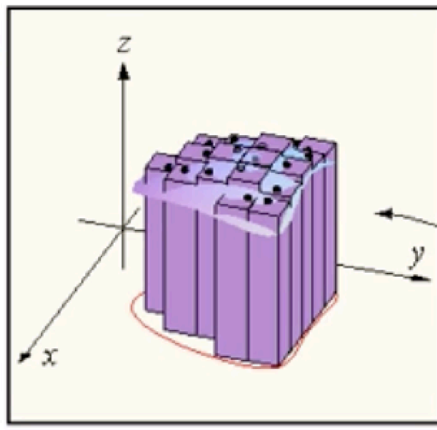
Finding Volume Under a Surface

Double Integral

Extending the concept of "area under the curve" to 3D, you can find the volume under a surface by dividing it into infinitely many small parallelepipeds (rectangular prisms) and summing up each one's volume.

Suppose R is a bounded region in the xy-plane, and $f(x,y)$ is a continuous non-negative function defined on R.

To define and compute the volume of this solid:



Draw a rectangle with the sides parallel to the coordinates axes, enclosing the region R.

Such a rectangle exists because R is bounded.

Using lines parallel to the coordinate axes, partition the rectangle into subrectangles.

Suppose there are n subrectangles, and the area of the kth subrectangle is ΔA_k.

For each k, choose a point (x_k^*, y_k^*) in the kth subrectangle.

If the dimensions of the kth subrectangle are small, then f does not vary much over this subrectangle.

Volume = (height) · (area of base) $= f(x_k^*, y_k^*) \cdot \Delta A_k$.

f is continuous.

$$\begin{pmatrix} \text{volume of the parallelepiped} \\ \text{over the } k\text{th subrectangle} \end{pmatrix} = f(x_k^*, y_k^*)\Delta A_k \approx \begin{pmatrix} \text{volume of the solid lying} \\ \text{under the graph of } f \text{ and} \\ \text{over the } k\text{th subrectangle} \end{pmatrix}.$$

$$\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \approx \begin{pmatrix} \text{volume of the solid lying under the} \\ \text{graph of } f \text{ and over the region } R \end{pmatrix}.$$

Definition: If f is a function of two variables and is continuous and non-negative on the region R in the xy-plane, then the **volume of the solid enclosed between the graph of f and the region R** is defined to be:

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \iint_R f(x,y)\,dA$$

where:

- R is the region in the xy-plane,

- $f(x, y)$ is the height of the surface at (x, y),
- dA represents an infinitesimal area element in R (e.g., $dx\,dy$).

Properties of Double Integrals

Theorem. If f is continuous on a region R (whose boundary is not too complicated), then

$$\iint\limits_R f(x,y)\,dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \text{ exists.}$$

Corollary. If the boundary of R is a smooth curve or a finite collection of such curves, and f is continuous on R, then $\iint\limits_R f(x,y)\,dA$ exists.

Theorem. Let f and g be functions defined on the region R, and let c be a constant. If the double integrals of f and g over R both exist, then

(1) $\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA.$

(2) $\iint\limits_R (f(x,y)+g(x,y))\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA.$

(3) $\iint\limits_R (f(x,y)-g(x,y))\,dA = \iint\limits_R f(x,y)\,dA - \iint\limits_R g(x,y)\,dA.$

(4) If R is composed of the regions R_1 and R_2, and the double integral of f exists over both R_1 and R_2, then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA.$$

- Additionally, the order of integration of the iterated integral is reversible so long as the function $f(x, y)$ is continuous over the region R.

Evaluating the Double Integral (Volume Under a Surface)

The volume V under a surface $z = f(x, y)$ over a region R in the xy-plane is given by:

$$V = \iint_R f(x, y)\,dA.$$

If the region R is rectangular, say $a \leq x \leq b$ and $c \leq y \leq d$, the double integral can be expressed as:

$$V = \int_a^b \left(\int_c^d f(x, y)\,dy\right) dx, = \int_c^d \left(\int_a^b f(x, y)\,dx\right) dy,$$

- Parentheses are used to show ordering, from inside to out.
- Graphically, the inner integral limits will extend from the one end of the strip to another. The outer integral limits will be the bounds in which these strips are summed over.

To solve the double integral, first integrate the inner integral with respect to x/y. Treat the other variable as a parameter (fixed/held constant). Then, take that expression and integrate it with respect to the second variable.

- After evaluating the inner definite integral, we will get an expression in terms of the other variable (to integrate with respect to).

Integration Techniques

1. Reversal of order of integration.
2. Factorize the integrand (use when $f(x,y) = g(x)h(y)$ and bounds are constant), which allows for separation of variables:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \left(\int_a^b g(x)\,dx\right) \cdot \left(\int_c^d h(y)\,dy\right)$$

3. Usage of symmetry to ease calculations when pluggin in bounds.

Generalization to Non-Rectangular Regions

Type I Region

- **Definition:** A type I region R is defined by the set: $R = \{(x,y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$
- **Condition:** If f is continuous on R, then the double integral of f over R is given by:

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Type II Region

- **Definition:** A type II region R is defined by the set: $R = \{(x,y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$
- **Condition:** If f is continuous on R, then the double integral of f over R is given by:

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Note: Whichever variable the bounds are is whichever variable the integration is done with respect to.

Reversing the Order of Integration & Changing the Bounds of Integration (Fubini's Theorem)

Reversing the order of integration in a double integral involves interchanging the roles of the variables in the integration process. This typically requires adjusting the bounds of integration to accurately reflect the new order.

- Imagine that the orientation of individual strips are rotated 90 degrees, and that they are summed across a different axis/direction.
- **Note**: the bounds are treated as $y = \dots$ when integrating with respect to y, and $x = \dots$ when integrating with respect to x.

Steps:

1. **Visualize the Region of Integration**:
   - Sketch the region of integration described by the current bounds.
   - Identify the relationships between x and y that define the region.
2. **Determine the New Bounds**:
   - Express the limits of one variable (e.g., x) as functions of the other variable (e.g., y).
   - Adjust the outer bounds to match the range of the independent variable in the new order.
3. **Rewrite the Integral**:
   - Swap the roles of dx and dy in the integral.
   - Update the bounds to reflect the new variable relationships.

Example: Reverse the order of integration for:

$$\int_0^1 \int_0^x f(x, y)\, dy\, dx$$

Step 1: Visualize the Region

- The region of integration is defined by:
  - $0 \leq x \leq 1$
  - $0 \leq y \leq x$
- This corresponds to the triangular region bounded by $y = 0$, $x = 1$, and $y = x$.

Step 2: Determine New Bounds

- Rewrite the region in terms of y:
  - $0 \leq y \leq 1$ (outer bounds for y)

- $0 \le x \le y$ (inner bounds for x)

Step 3: Rewrite the Integral The new integral becomes:

$$\int_0^1 \int_0^y f(x,y)\,dx\,dy$$

Polar Double Integrals

Introduction

Polar regions refer to the area between two rays $\theta = \alpha$ and $\theta = \beta$.

$$\int_\alpha^\beta \int_{r(\theta_1)}^{r(\theta_2)} f(r,\theta) \ \cdot rdrd\theta$$

Evaluating Polar Double Integrals

1. Convert the function integrand into polar form: $x = rcos(\theta), y = rsin(\theta)$. Convert the height function into polar form as well.
2. Adjust the area element: Replace dA with $r\,dr\,d\theta$.
3. Describe the region of integration R in terms of polar coordinates, determining the bounds of integration with r and θ.
4. Evaluate the iterated integral, typically integrating with respect to r first and then θ.

Misc

Mean Value Theorem in 3D

The average value of a function $f(x,y)$ over a region R is given by:

$$\text{Average Value} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

Where:

- $A(R)$ is the area of the region R.
- $\iint_R f(x,y)\,dA$ is the double integral of $f(x,y)$ over R, representing the volume under the surface $z = f(x,y)$ and above the region R.

Finding the Area of a Region

Double integrals are useful for finding the area of irregular or nonrectangular regions in the plane. This approach involves integrating over a specified region, using the fact that the integral of 1 over a region gives its area.

- **Region of Integration**: A nonrectangular region in the plane can be defined by a combination of boundary equations or inequalities.
- **Double Integral**: The area of a region R is given by:

$$\text{Area} = \iint_R 1 \, dA$$

2. Integrating Parametrized Functions

Parametric Representations of Surfaces

Definition and Properties

Just as $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ describes a curve in 3-space, $\vec{r}(u,v) = \langle (x(u,v), y(u,v), z(u,v) \rangle$ describes a **surface in 3-space** as a vector-valued function of **two variables**.

The **vector-valued function** $\vec{r}(u,v) = \langle (x(u,v), y(u,v), z(u,v) \rangle$ is said to be **continuous** if each of its component functions ($x(u,v)$, $y(u,v)$, and $z(u,v)$) is continuous.

Partial Derivatives

If the vector-valued function $\vec{r}(u,v) = \langle (x(u,v), y(u,v), z(u,v) \rangle$, the **partial derivatives** of r are defined as follows:

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

Both $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are 3-space vector-valued functions of two variables.

An alternative notation is:

- $\frac{\partial r}{\partial u} = \mathbf{r}_u$
- $\frac{\partial r}{\partial v} = \mathbf{r}_v$

Tangent Vector to the Constant Curve (Application of Partial Derivative)

Suppose the vector-valued function is given by:

$$\vec{r}(u,v) = \langle (x(u,v), y(u,v), z(u,v) \rangle$$

When v is fixed at (v_0), the constant v-curve is parameterized as:

$$r(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

The tangent vector to this curve at (u_0, v_0) is given by:

$$\frac{d}{du}\Big[r(u, v_0)\Big]_{u=u_0} = \mathbf{r}_u(u_0, v_0)$$

- We see how the function changes as v is held constant (thus the partial derivative with respect to u). It is then evaluated at u_0.

Note: $\mathbf{r}_u(u_0, v_0)$ is the tangent vector to a constant v-curve at point (u_0, v_0), and the opposite is true for $\mathbf{r}_v(u_0, v_0)$.

Determining the Equation of the Tangent Plane

A **tangent plane** H to a surface S at a point P is a plane that contains all tangent lines to smooth curves on S passing through P.

Let $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ parametrize a surface S. If:

- (x_0, y_0, z_0) is a point on S corresponding to the input parameter (u_0, v_0), and
- $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \neq 0$, then H, the tangent plane to S at (x_0, y_0, z_0), has the equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

1. **Observation**: $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ are tangent vectors to smooth curves on S passing through (x_0, y_0, z_0).
   - If these vectors are non-zero, they must lie in the plane H.
2. **Normal Vector**:
   - The cross product $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \neq 0$ defines a normal vector to H.
3. **Point-Normal Form**:
   - Since H passes through (x_0, y_0, z_0) and has a normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, it can be expressed in **point-normal form**:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Technical Remarks on Tangent Planes for Parametric Surfaces

1. **Non-existence**:
   - For some surfaces, there are points where no tangent plane exists (e.g., corner/cusp).
   - At such points, no plane contains all tangent lines to all smooth curves passing through that point.
2. **Non-uniqueness**:
   - For some surfaces, there are points where multiple tangent planes exist.
3. **Existence and Uniqueness for Differentiable Functions**:

- If $f(x, y)$ is differentiable at (x_0, y_0), then the graph of f has a unique tangent plane at $(x_0, y_0, f(x_0, y_0))$.

4. **Uniqueness for Parametric Surfaces**:
   - If $\mathbf{r}(u, v)$ parametrizes S, with $\mathbf{r}(u_0, v_0) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, and $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ are non-zero and non-parallel vectors, then:
     - There can be at most one tangent plane to S at (x_0, y_0, z_0).

Principle Unit Normal Vector

Suppose $\mathbf{r}(u, v)$ parametrizes the surface S, and the point (x_0, y_0, z_0) on S corresponds to the parameter (u_0, v_0).

If $\mathbf{r}_u$ and $\mathbf{r}_v$ exist at (u_0, v_0) and $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) \neq 0$, then the **principal unit normal vector** to S at (x_0, y_0, z_0), denoted $\mathbf{n}(u_0, v_0)$, is given by:

$$\mathbf{n}(u_0, v_0) = \frac{\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)}{\|\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)\|}$$

Strategies for Parametrization & Eliminating Parameters

Surface Area Over a Region

Previously, we had only discussed the surface area of a surface of revolution:

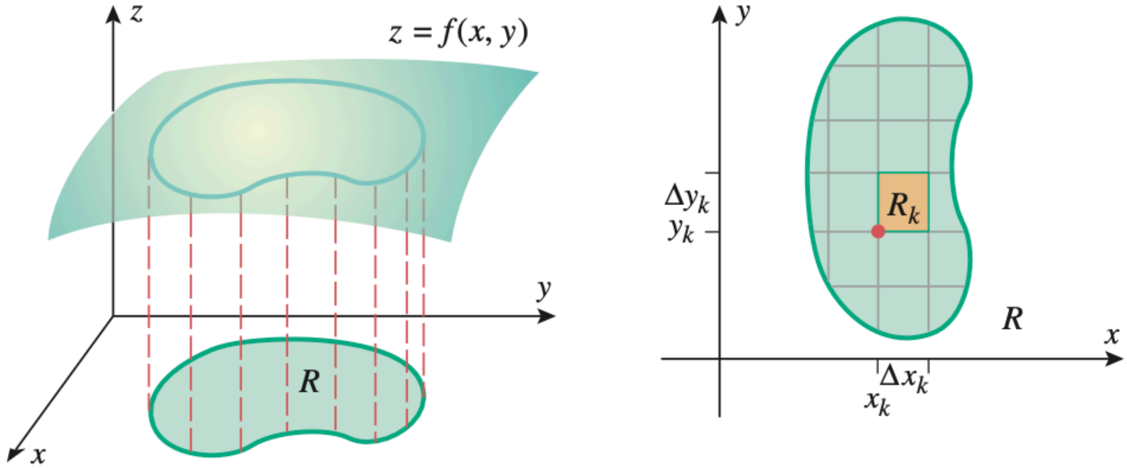
$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Surface Area: Form of z = f(x,y)

Now, we move on more generally to consider a surface σ of the form $z = f(x, y)$ defined over a region R in the xy-plane.

- Assume that f has continuous first partial derivatives at the interior points of R. This ensures the surface will have a nonvertical tangent plane at each interior point of R.
- We subdivide R into rectangular regions by lines parallel to the x and y axes and discard any nonrectangular portions that contain points on the boundary of R.

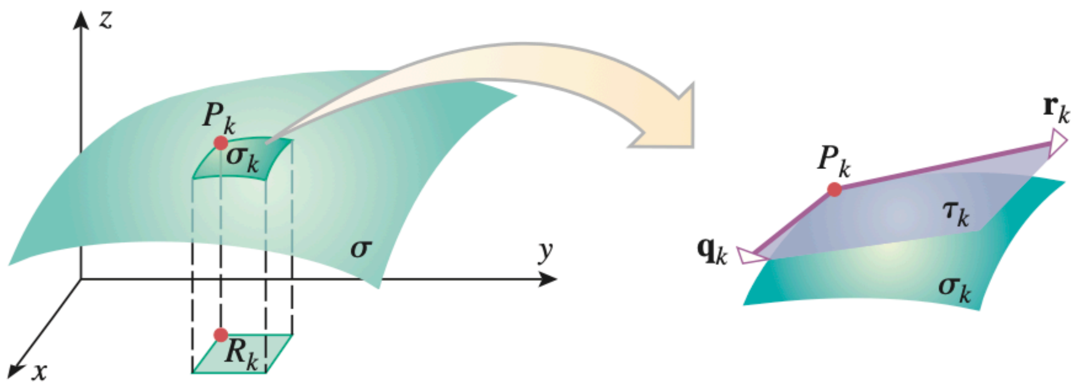
- Assume there are n rectangles labeled $R_1, R_2, \ldots, R_n$.



(a) (b)

Let (x_k, y_k) be the lower-left corner of the kth rectangle R_k, and assume that R_k has area $\Delta A_k = \Delta x_k \Delta y_k$, where Δx_k and Δy_k are the dimensions of R_k.

- The portion of σ that lies over R_k will be some *curvilinear patch* on the surface that has a corner at $P_k(x_k, y_k, f(x_k, y_k))$; denote the area of this patch by ΔS_k.



(a) (b)

To obtain an approximation of ΔS_k, we will replace σ by the tangent plane to σ at P_k. The equation of this tangent plane is:

$$z = f(x_k, y_k) + f_x(x_k, y_k)(x - x_k) + f_y(x_k, y_k)(y - y_k)$$

The portion of the tangent plane that lies over R_k will be a parallelogram τ_k. This parallelogram will have a vertex at P_k and adjacent sides determined by the vectors:

$$\mathbf{q}_k = \left(\Delta x_k, 0, \frac{\partial z}{\partial x}\Delta x_k\right) \quad \text{and} \quad \mathbf{r}_k = \left(0, \Delta y_k, \frac{\partial z}{\partial y}\Delta y_k\right)$$

- If the dimensions of R_k are small, then τ_k should provide a good approximation to the curvilinear patch σ_k.
- The area of the parallelogram τ_k is the length of the cross product of $\mathbf{q}_k$ and $\mathbf{r}_k$. Thus, we expect the approximation

  $$\Delta S_k \approx \text{area } \tau_k = \|\mathbf{q}_k \times \mathbf{r}_k\|$$

  ```
  to be good when $\Delta x_k$ and $\Delta y_k$ are close to 0.
  ```

Computing the cross product yields

$$\|\mathbf{q}_k \times \mathbf{r}_k\| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_k & 0 & \frac{\partial z}{\partial x}\Delta x_k \\ 0 & \Delta y_k & \frac{\partial z}{\partial y}\Delta y_k \end{vmatrix} \right\| = \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right)\Delta x_k \Delta y_k$$

$$\Delta S_k \approx \left\| \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right)\Delta x_k \Delta y_k \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A_k$$

It follows that the entire surface area can be approximated as

$$S \approx \sum_{k=1}^{n} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A_k$$

- If we assume that the errors in the approximations approach zero as n increases in such a way that the dimensions of the rectangles approach zero, then it is plausible that the exact value of S is

$$S = \lim_{n \to +\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A_k$$

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

Problem-Solving Procedure:

1. Find an expression in the form of $z = f(x, y)$. Then, calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
2. Understand the 2D region/area that small elements of the surface is being summed over. Set these as the bounds for the double integral.

Surface Area: Parametric Form

Let σ be a smooth parametric surface.

- This surface is parametrized by the vector-valued function $\mathbf{r}(u, v)$.
- The domain of $\mathbf{r}(u, v)$ is denoted by D.

The surface area, S, of σ is defined by the double integral:

$$S = \iint_D ||\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)||dA$$

- Bounds of the iterated integral is the 2D region u/v that the surface is over. The order doesn't matter according to Fubini's Theorem, which states that the order of integration can be interchanged since it is a continuous function over a rectangular domain.
- Understand the surface as being divided up into tiny parallelogram approximations, whose individual areas are found with the cross product and summed over the domain.

2. **Explicit Form as a Special Case**

- If the surface is given explicitly as $z = f(x, y)$, we can express it parametrically as:

$$\mathbf{r}(x, y) = (x, y, f(x, y))$$

- Taking partial derivatives:

$$\mathbf{r}_x = (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y)$$

- Computing the cross product:

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1)$$

- Taking its magnitude:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{1 + f_x^2 + f_y^2}$$

- Thus, substituting this into the parametric formula recovers the explicit function formula.

3. Triple Integrals

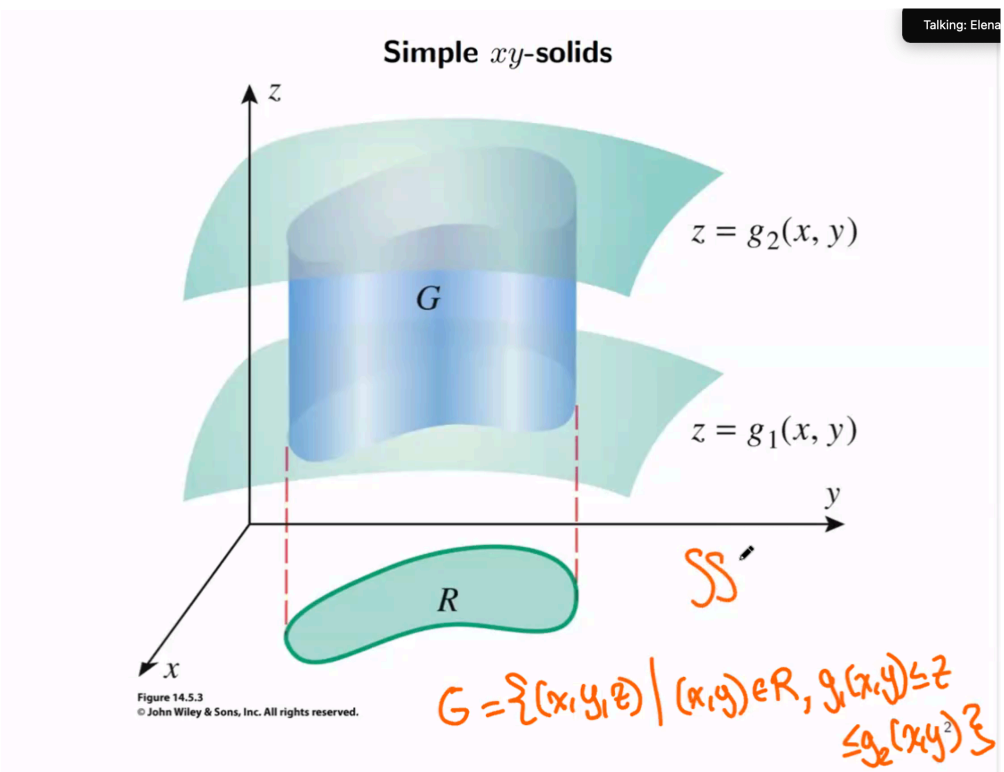
Triple Integrals

Definition

A **triple integral** extends the concept of a double integral to three dimensions, integrating a function $f(x, y, z)$ over a 3D region E.

The triple integral is used to find the:

- **Volume** of a 3D region: $\iiint_E dV$.
- **Mass** of an object with **variable density** $\rho(x, y, z)$: $\iiint_E \rho(x, y, z)\, dV$.
- **Average value** of a function over a 3D region.



1. **Linearity**:
   - $\iiint_E [af(x, y, z) + bg(x, y, z)]\, dV = a \iiint_E f\, dV + b \iiint_E g\, dV$ for constants a, b.
2. **Additivity**:
   - If $E = E_1 \cup E_2$ (disjoint regions), then $\iiint_E f\, dV = \iiint_{E_1} f\, dV + \iiint_{E_2} f\, dV$.
3. **Comparison**:
   - If $f(x, y, z) \le g(x, y, z)$ over E, then $\iiint_E f\, dV \le \iiint_E g\, dV$.

Triple Integrals over General Regions

Let G be a **simple xy-solid** defined as follows:

- **Upper surface**: $z = g_2(x, y)$
- **Lower surface**: $z = g_1(x, y)$
- **Projection onto the xy-plane**: A region R.

If $f(x, y, z)$ is continuous on G, then:

$$\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz \right] dA$$

Think of the **inner limits** to be from **surface to surface**, **middle limits** to be from **curve to curve**, and **outer limits** to be point to point.

- First identify the **surface to surface** for the innermost integral. Travelling down, collapse into a 2-D region R.
- Adjust the constraints from the equation accordingly (e.g., for a projection down onto the xy plane, set $z = 0$ in the equation to see the relationship between x and y).
- Continue solving as you would for a double integral.

Changing the Order of Integration (Fubini's Theorem)

Constant Bounds, Rectangular Projection Region When all the limits of the region are constants, the integration region is a rectangular box and the order of integration can be changed freely without adjusting the bounds.

Changing the Order of Integration for General Cases If the limits are **not constants**, then at least one limit is dependent on another variable.

- The innermost bounds must be a function of the outer two variables (e.g., if dz is on the inside, then the bounds will be a function of x and y).
- The middle bound will be a function of one variable, which is the outermost variable.
- The outermost bounds must become constants.

We should seek to make the inner integrals as simple as possible since the outermost integral is guarenteed to have constant bounds. For **triple integrals**, there are 6 possible permutations for the order of integration (3 variables).

There are 3 types of solids: simple xy solids, simple yz solids, and simple xz solids. This means that the projection of the solid onto their respective planes results in a simple, well-defined shape.

For easy integration, if the bounds result in a **simple projection on the xy plane**, it is a simple xy solid and z should be the innermost variable to be integrated first. The same concept applies to the 2 others.

Product Property for Triple Integrals Over a Rectangular Box

The **product property** states that if a function $f(x, y, z)$ can be expressed as the product of three single-variable functions, i.e., $f(x, y, z) = f_1(x) \cdot f_2(y) \cdot f_3(z)$, then the triple integral over a rectangular box B simplifies to the product of three separate single-variable integrals.

- Let $B = [a, b] \times [c, d] \times [e, g]$ be a rectangular box. For $f(x, y, z) = f_1(x)f_2(y)f_3(z)$:

$$\iiint_B f(x, y, z)\, dV = \left(\int_a^b f_1(x)\, dx\right)\left(\int_c^d f_2(y)\, dy\right)\left(\int_e^g f_3(z)\, dz\right)$$

Conditions for Application:

1. **Rectangular Domain**: The region of integration must be a rectangular box.
2. **Separability**: The integrand must be expressible as a product of functions, each depending on only one variable (x, y, or z).
3. **Integrability**: $f_1(x)$, $f_2(y)$, and $f_3(z)$ must be integrable over their respective intervals.

Applications of Multiple Integrals

Mass and Density of a Lamina

Definition and Properties

- A **lamina** is a thin, flat two-dimensional surface with mass but negligible thickness.
- It is characterized by a **density function** $\sigma(x, y)$, which represents the mass per unit area at a point (x, y).
  - Homogeneous when composition is uniform; otherwise inhomogeneous.

Total Mass

- The total mass M of the lamina is determined by integrating the density function over the region R occupied by the lamina:

$$M = \iint_R \sigma(x, y)\, dA$$

Center of Mass/Gravity

Definition and Properties

- The **center of gravity** (or center of mass) of an object is the point where the entire weight of the object is considered to be concentrated.
- For any object with a continuous mass distribution, it is computed as a weighted average of the positions of all the mass elements.

For objects located at discrete locations:

If a system consists of **n** particles, each with mass m_i at position (x_i, y_i, z_i), the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

where:

- m_i is the mass of the i-th particle.
- x_i, y_i, z_i are the coordinates of the i-th particle.

In other words, for each individual direction it's still a weighted average where it's the $\frac{\text{Mass}}{\text{Mass of System}} \times \text{Position.}$

In general, if a body occupies a region D with density function $\rho(\mathbf{r})$, the center of gravity $\bar{\mathbf{r}}$ is given by:

$$\bar{\mathbf{r}} = \frac{\iiint_D \mathbf{r}\,\rho(\mathbf{r})\,dV}{\iiint_D \rho(\mathbf{r})\,dV}$$

Center of Mass in Three Dimensions (Solid Body)

For a solid occupying a volume V with a density function $\rho(x, y, z)$, the center of mass is:

$$\bar{x} = \frac{\iiint_V x\rho(x, y, z)\,dV}{\iiint_V \rho(x, y, z)\,dV}$$

$$\bar{y} = \frac{\iiint_V y\rho(x, y, z)\,dV}{\iiint_V \rho(x, y, z)\,dV}$$

$$\bar{z} = \frac{\iiint_V z\rho(x, y, z)\,dV}{\iiint_V \rho(x, y, z)\,dV}$$

Center of Gravity for 2D Lamina

Given the density function $\sigma(x, y)$ over the region R, the coordinates of the center of gravity $(\bar{x}, \bar{y})$ are:

$$\bar{x} = \frac{\iint_R x\,\sigma(x, y)\,dA}{\iint_R \sigma(x, y)\,dA}$$

$$\bar{y} = \frac{\iint_R y\,\sigma(x, y)\,dA}{\iint_R \sigma(x, y)\,dA}$$

- The numerator in each formula represents the moment of the mass distribution about the corresponding axis.
- The denominator is the total mass of the lamina.
- The fraction represents a **weighted average**, with the weight being the **density at that specific point in the region**

Pappus' Theorem and The Centroid

The **centroid** of a geometric object is the average position of all the points in the shape. It is the "geometric center" and is where the shape would balance perfectly if if had uniform density.

Pappus' Centroid Theorem relates the surface area and volume of a solid of revolution to the distance traveled by its centroid during rotation.

- It is particularly useful for computing volumes and surface areas without performing complex integrations.

Theorem Statements

First Theorem (Volume):

- The volume V of a solid generated by rotating a plane region of area A about an external axis (in the same plane) is:

$$V = A \cdot d$$

```
where $d$ is the distance traveled by the centroid of the region,
typically given by $d = 2\pi R$, with $R$ being the distance from the
centroid to the axis of rotation.
```

Second Theorem (Surface Area):

- The surface area S of a solid of revolution obtained by rotating a curve of length L about an external axis is:

$$S = L \cdot d$$

```
with $d$ defined similarly as the distance traveled by the centroid of
the curve.
```

3-D Center of Gravity

Definition for Three-Dimensional Bodies

- For a solid occupying a region V with a density function $\rho(x, y, z)$, the center of gravity (or center of mass) is calculated using triple integrals.

Coordinate Formulas

The coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the center of gravity are:

$$\bar{x} = \frac{\iiint_V x\,\rho(x, y, z)\,dV}{\iiint_V \rho(x, y, z)\,dV}$$

$$\bar{y} = \frac{\iiint_V y\,\rho(x, y, z)\,dV}{\iiint_V \rho(x, y, z)\,dV}$$

$$\bar{z} = \frac{\iiint_V z\,\rho(x, y, z)\,dV}{\iiint_V \rho(x, y, z)\,dV}$$

Determining the Location of Centroid

The coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the centroid of a 3D shape are:

$$\bar{x} = \frac{\iiint_V x\,dV}{\iiint_V dV}$$

$$\bar{y} = \frac{\iiint_V y\,dV}{\iiint_V dV}$$

$$\bar{z} = \frac{\iiint_V z\,dV}{\iiint_V dV}$$

Note: The centroid is at the same location as the center of gravity **only** when density is uniform. Even when the density of the object is not uniform, it is still calculated using the same formula as it is a geometric property only.

Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) extend polar coordinates into three dimensions by adding the height z.

- r: The radial distance from the z-axis.
- θ: The angle measured counterclockwise from the positive x-axis in the xy-plane.
- z: The same as in Cartesian coordinates, representing height.

A triple integral in cylindrical coordinates is written as:

$$\int\int\int_{\text{Region}} f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

where:

- $r_1 \leq r \leq r_2$ represents the radial bounds,
- $\theta_1 \leq \theta \leq \theta_2$ represents the angular bounds,
- $z_1 \leq z \leq z_2$ represents the height bounds.
- The volume element in cylindrical coordinates is given by: $dV = r\, dr\, d\theta dz$

Triple Integrals in Spherical Coordinates

Spherical coordinates (ρ, θ, ϕ) describe a point in 3D space based on its radial distance and two angles:

- ρ: The radial distance from the origin.
- θ: The azimuthal angle (same as in cylindrical coordinates), measured counterclockwise from the positive x-axis.
- ϕ: The polar angle, measured from the positive z-axis.

The conversion formulas between Cartesian and spherical coordinates are:

- $x = \rho \sin\phi \cos\theta$
- $y = \rho \sin\phi \sin\theta$
- $z = \rho \cos\phi$

A triple integral in spherical coordinates is written as:

$$\int\int\int_{\text{Region}} f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

where:

- $\rho_1 \leq \rho \leq \rho_2$ represents the radial bounds,
- $\phi_1 \leq \phi \leq \phi_2$ represents the polar angle bounds,
- $\theta_1 \leq \theta \leq \theta_2$ represents the azimuthal angle bounds.
- The volume element in spherical coordinates is: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Coordinate System	**Variables**	**Volume Element**
Cartesian	(x, y, z)	$dx \, dy \, dz$
Cylindrical	(r, θ, z)	$r \, dr \, d\theta \, dz$
Spherical	(ρ, θ, ϕ)	$\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Problem-Solving Procedure

1. Given two surfaces, set them equal to one another to determine curve of intersection. If this curve of intersection maps neatly onto a specific plane, proceed to determine appropriate boundaries of integration region.
2. Determine whether there needs to be a change of coordinate systems.
3. Parametrize the curve and then integrate via change of variables/Jacobian.

4. Change of Variables

Change of Variables in Multiple Integrals

Introduction

In single variable calculus, the **substitution rule for definite integrals** states that if $f(x)$ is continuous on $[a, b]$, and we define a new variable:

- $x = g(u)$, where g is differentiable,
- $g(c) = a$ and $g(d) = b$,

then the integral transforms as:

$$\int_c^d f(g(u))g'(u)\,du = \int_a^b f(x)\,dx$$

where $g'(u)$ is the transformation factor.

In single variable calculus, the common integration technique of **u-substitution** is the result of going from left >>> right.

However, some integrals may actually become easier going right >>> left. For example, $\int \sqrt{1-x^2}\,dx$ is complex, but if we used $x = \sin(u)$, then the integral becomes $\int \cos^2(u)\,du$, which is much simpler.

This same principle motivates change of variables in multivariable calculus.

The Jacobian Matrix and Determinant

Purpose of the Jacobian

- The Jacobian determinant acts as a scaling factor.
- It ensures that the infinitesimal area (or volume) element is correctly transformed from one coordinate system to another.
- It helps preserve the integral's value under the transformation.

Definition of the Jacobian Matrix and Determinant The **Jacobian matrix** is a matrix whose entries are the partial derivatives of the **new** coordinates with respect to the **old** coordinates.

For a transformation T given by $x = g(u, v)$ and $y = h(u, v)$, the Jacobian determinant is:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

For three variables, $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$, the Jacobian determinant is:

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

A Note Regarding Notation:

- $J(u,v)$ refers to the scalar multiplier necessary to transform **from** (u,v) **to** (x,y).
  - $T(x,y)$ refers to the transformation applied **onto** (x,y), mapping it to a **new** coordinate system: $T(x,y) = (u(x,y), v(x,y))$.
- In partial form, $\frac{\partial(x,y)}{\partial(u,v)}$ holds the **new** coordinates on **top** and **old on bottom**.
- Each entry in the determinant will also have the partials of **new w.r.t. old**.

Properties of the Jacobian Determinant Due to the nature of the process, the Jacobian determinants going either direction in the transformation are **inverses** of one another:

$$J(u,v) = \frac{1}{J(x,y)}$$

- This is particularly **helpful** when you have **u/v defined in x/y** and need to find the Jacobian determinant for transformation, but **don't want to rearrange to have x/y defined in u/v** (also works the other way around).

Evaluating Multiple Integrals via Change of Variables

The general problem-solving procedure is:

1. Identify the transformation and new bounds for the transformed region (make an appropriate transformation based on integrand + boundary conditions).
2. Calculate the determinant of the Jacobian matrix. Use inverse property if needed.
3. Transform original integrand and multiply with scaling factor. Evaluate the multiple integral.

Double Integrals

- Suppose we have a transformation T from the uv-plane to the xy-plane, defined by: $x = g(u,v)$ and $y = h(u,v)$.
- The double integral of a function $f(x,y)$ over a region R in the xy-plane can be transformed into a double integral over a region S in the uv-plane as follows:

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

- The absolute value of the Jacobian determinant ensures the area element is positive.

Note: Change the original integrand to be in the correct terms **in addition to** the scalar transformation multiplier.

Triple Integrals

- For a transformation T from uvw-space to xyz-space, defined by: $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$.
- The triple integral of a function $f(x,y,z)$ over a region R in xyz-space can be transformed into a triple integral over a region S in uvw-space:

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(g(u,v,w),h(u,v,w),k(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

- The absolute value of the Jacobian determinant ensures the volume element is positive.

Multiple Transformations (e.g., $(x,y) \to (u,v) \to (r,\theta)$) General Concept:

- Start with an initial coordinate system, e.g., (x,y).
- Apply a transformation to an intermediate coordinate system, e.g., (u,v).
- Apply another transformation to a final coordinate system, e.g., (r,θ).
- Each transformation has its own Jacobian determinant. Applying a sequence of transformations will ultimately lead to a simpler region or integrand.

Example Problem: Evaluate the integral

$$\iint_R \sqrt{16x^2 + 9y^2}\,dA$$

where R is the region enclosed by the ellipse $(x^2/9) + (y^2/16) = 1$.

The ellipse equation suggests the transformation:

- $x = 3u$ (because $x^2/9 = u^2$)
- $y = 4v$ (because $y^2/16 = v^2$) This transforms the ellipse $(x^2/9) + (y^2/16) = 1$ into $u^2 + v^2 = 1$, which is the unit circle.

Now, we compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = (3)(4) - (0)(0) = 12$$

Next, we transform the integrand:

$$\sqrt{16x^2 + 9y^2} = \sqrt{16(3u)^2 + 9(4v)^2} = \sqrt{144u^2 + 144v^2} = \sqrt{144(u^2 + v^2)} = 12\sqrt{u^2 + v^2}$$

Now we have the integral in the uv-plane:

$$\iint_R \sqrt{16x^2 + 9y^2}\, dA = \iint_S 12\sqrt{u^2 + v^2} \cdot 12\, du\, dv = 144 \iint_S \sqrt{u^2 + v^2}\, du\, dv$$

where S is the unit circle $u^2 + v^2 \leq 1$.

Now, we switch to polar coordinates in the uv-plane:

- $u = r\cos\theta$
- $v = r\sin\theta$
- $u^2 + v^2 = r^2$
- $du\, dv = r\, dr\, d\theta$

The region S in polar coordinates is $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. The integrand $\sqrt{u^2 + v^2}$ becomes r. The integral becomes:

$$144 \iint_S \sqrt{u^2 + v^2}\, du\, dv = 144 \int_0^{2\pi} \int_0^1 r \cdot r\, dr\, d\theta = 144 \int_0^{2\pi} \int_0^1 r^2\, dr\, d\theta$$

Evaluate the integral:

$$144 \int_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^1 d\theta = 144 \int_0^{2\pi} \frac{1}{3}\, d\theta = 48 \int_0^{2\pi} d\theta$$
$$= 48[\theta]_0^{2\pi} = 48(2\pi - 0) = 96\pi$$

Final Answer: The value of the integral is 96π.

Understanding Polar, Cylindrical, and Spherical Integrand Multipliers

Polar Coordinates

- Transformation: $x = r\cos(\theta)$, $y = r\sin(\theta)$.
- Jacobian:

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r(\cos^2(\theta) + \sin^2(\theta)) = r$$

Cylindrical Coordinates

- Transformation: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$.
- Jacobian:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Spherical Coordinates

- Transformation: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.
- Jacobian:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{vmatrix} = \rho^2\sin(\phi)$$

If T has vector form $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}$, then:

$$|J(u,v)| = \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|$$

where $\mathbf{r}_u(u,v)$ and $\mathbf{r}_v(u,v)$ are viewed as vectors in 3-space with third component 0.

Finding the Outputs of a Transformation

If given a graph of (u,v) and the equations $x = f(u,v)$ and $y = g(u,v)$, how does one find the new region S ?

Moving along the regions for the original (u,v) coordinates, (u,v) must follow a certain conditions (e.g., $u = 1$ while v varies from $0 \le v \le 1$). Taking advantage of this, rewrite the equations for x and y using these limiting conditions moving along specific boundaries.

Example Problem: Sketch the image in the xy-plane of the set S under the given transformation. S is the rectangle defined by $1 \le u \le 2$ and $0 \le v \le \pi/2$. The transformation is:

- $x = u\cos v$
- $y = u\sin v$

Solution: We'll examine the images of the four sides of the rectangle S in the xy-plane:

- **Side 1:** $u = 1$, $0 \le v \le \pi/2$
- **Side 2:** $u = 2$, $0 \le v \le \pi/2$
- **Side 3:** $v = 0$, $1 \le u \le 2$
- **Side 4:** $v = \pi/2$, $1 \le u \le 2$

Transform each side:

- **Side 1 (u = 1):** $x = 1 \cdot \cos v = \cos v$ $y = 1 \cdot \sin v = \sin v$ Since $0 \le v \le \pi/2$, this represents a quarter-circle of radius 1 in the first quadrant, from (1, 0) to (0, 1). We can recognize this as $x^2 + y^2 = 1$ with $x \geq 0$ and $y \geq 0$.

- **Side 2 (u = 2):** $x = 2\cos v$ $y = 2\sin v$ Since $0 \leq v \leq \pi/2$, this is a quarter-circle of radius 2 in the first quadrant, from (2, 0) to (0, 2). This is $x^2 + y^2 = 4$ with $x \geq 0$ and $y \geq 0$.
- **Side 3 (v = 0):** $x = u\cos 0 = u$ $y = u\sin 0 = 0$ Since $1 \leq u \leq 2$, this is a line segment on the x-axis from $x = 1$ to $x = 2$.
- **Side 4 (v = π/2):** $x = u\cos(\pi/2) = 0$ $y = u\sin(\pi/2) = u$ Since $1 \leq u \leq 2$, this is a line segment on the y-axis from $y = 1$ to $y = 2$.

Sketch: The image is the region bounded by: 4. A quarter-circle of radius 1 in the first quadrant (Side 1). 5. A quarter-circle of radius 2 in the first quadrant (Side 2). 6. A line segment on the x-axis from $x = 1$ to $x = 2$ (Side 3). 7. A line segment on the y-axis from $y = 1$ to $y = 2$ (Side 4).

The region is a quarter-annulus (a quarter of the region between two concentric circles). **Final Answer:** The image is the region in the first quadrant between the quarter-circles of radius 1 and 2, bounded by the x and y axes.

1. Vector Fields

Vector Fields

A vector field is a fundamental concept in vector calculus that **assigns a vector to each point in a subset of space**. It's like having an arrow at every point, where each arrow has a specific magnitude and direction.

Definition

- Formally, a vector field on a region D in $\mathbb{R}^2$ (2-dimensional space) is a function $\vec{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{F}(x, y)$.
- Similarly, a vector field on a region E in $\mathbb{R}^3$ (3-dimensional space) is a function $\vec{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{F}(x, y, z)$.

Representation/Notation

In $\mathbb{R}^2$, a vector field is typically represented as:

$$\vec{F}(x, y) = F_x(x, y)\hat{\imath} + F_y(x, y)\hat{\jmath}$$

where $F_x(x, y)$ and $F_y(x, y)$ are scalar functions representing the x and y components of the vector field, respectively, and $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors in the x and y directions.

In $\mathbb{R}^3$, a vector field is typically represented as:

$$\vec{F}(x, y, z) = F_x(x, y, z)\hat{\imath} + F_y(x, y, z)\hat{\jmath} + F_z(x, y, z)\hat{k}$$

where $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$ are scalar functions representing the x, y, and z components of the vector field, respectively, and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit vectors in the x, y, and z directions.

Del and Other Related Operators

Definition and Properties

The Del operator, denoted by ∇, is a vector differential operator used in vector calculus. It can be thought of as a vector of partial derivative operators. In three-dimensional Cartesian coordinates, it is defined as:

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit vectors in the x, y, and z directions, respectively.

Note that it is

Property: Linearity

- For any scalar functions f and g and constants a and b, the Del operator satisfies the property of linearity.

$$\nabla(af + bg) = a\nabla f + b\nabla g$$

Divergence (Scalar Output)

The divergence of a vector field $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is a scalar function that measures the "outwardness" of the vector field at a given point. It quantifies the magnitude of the vector field's source or sink at that point. The divergence is defined as:

$$\text{div}\ \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

- If $\nabla \cdot \vec{F} > 0$, the vector field has a net outward flow at that point, acting as a source.
- If $\nabla \cdot \vec{F} < 0$, the vector field has a net inward flow at that point, acting as a sink.
- If $\nabla \cdot \vec{F} = 0$, the vector field is said to be solenoidal or divergence-free, indicating that there is no net flow in or out of that point.

Curl (Vector Output)

The curl of a vector field $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is a vector function that describes the infinitesimal rotation of the vector field at a given point. It is defined as:

$$\text{curl}\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

- The direction of $\nabla \times \vec{F}$ indicates the axis of rotation of the vector field at that point.
- The magnitude of $\nabla \times \vec{F}$ represents the magnitude of the rotation.
- If $\nabla \times \vec{F} = \vec{0}$, the vector field is said to be irrotational, meaning there is no local rotation at that point.

Laplacian (Scalar Output)

The Laplacian is the **divergence of the gradient**. It measures the difference between the average value of a field in an infinitesimal neighborhood around a point and the value of the field at that point.

For a scalar function $\phi(x, y, z)$, the Laplacian is a scalar function defined as:

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

- In order words, it is the sum of the second partials with respect to each variable in $\phi(x, y, z)$.

The Laplacian of a vector function $\vec{F}(x, y, z)$ is defined as:

$$\nabla^2\vec{F} = \frac{\partial^2 F_x}{\partial x^2}\hat{i} + \frac{\partial^2 F_y}{\partial y^2}\hat{j} + \frac{\partial^2 F_z}{\partial z^2}\hat{k}$$

- In other words, it is the sum of second partials for each component of the vector function with respect to that component.

Interpretations:

- If $\nabla^2\phi > 0$ at a point, the value of ϕ at that point is **less than** the *average* value of ϕ in the surrounding infinitesimal region.
- If $\nabla^2\phi < 0$ at a point, the value of ϕ at that point is **greater than** the *average* value of ϕ in the surrounding infinitesimal region.
- If $\nabla^2\phi = 0$, the value of the function at that point is approximately equal to the *average* value of the ϕ in the neighborhood.
- **Note**: Unlike the second derivative/partials test, the Laplacian is a differential operator that provides information about the function's shape *but does not necessarily determine the location of local maxima/minima*.

Additional Notes

- **Divergence** and **curl** are intrinsic properties of a vector field. This means that **regardless of the coordinate system**, the computation of div and curl will return the same results.
- The Laplacian of a vector field is related to the gradient and divergence through the vector identity:

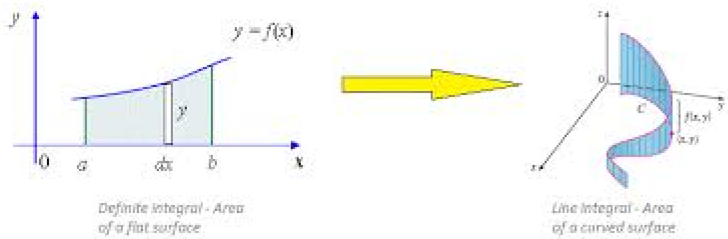
$$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$$

2. Line Integrals

Line Integral of a Scalar Field

A **line integral** is a type of integral where a function is evaluated along a curve rather than over an interval or region.

A **scalar line integral**, geometrically, is the **area of the sheet** formed by the height of the scalar field function along a curve C.



Definition and Properties

The line integral of a scalar function $f(x, y, z)$ along a curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ with respect to arc length is given by:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, |\mathbf{r}'(t)|\, dt$$

where $|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$ is the magnitude of the derivative of the position vector.

- $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

Properties:

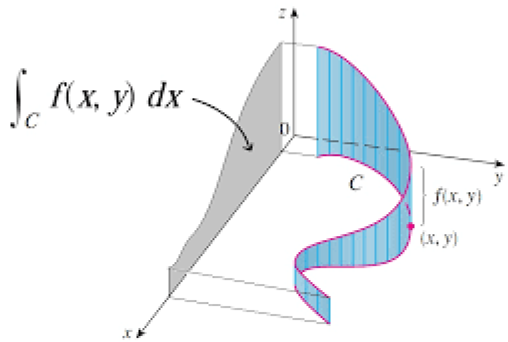
- $\int_C (f + g)\, ds = \int_C f\, ds + \int_C g\, ds.$
- $\int_C k \cdot f\, ds = k \int_C f\, ds$, where k is a constant.
- If a curve C is composed of two subcurves C_1 and C_2, then $\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds.$

Evaluating a Line Integral

1. Determine an appropriate parametrization the curve. Express x, y, and z in terms of t, with bounds for t.
2. Use the parametrization to express the integrand and differential in terms of t and dt.
3. Evaluate the line integral.

Line Integrals with Respect to x, y, and z

Line integrals can also be defined with respect to x, y, or z. These sum up elements of the



sheet **in a single direction**.

The line integral with respect to x is given by: $\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t)) \frac{dx}{dt}\, dt$. The line integral with respect to y is: $\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t)) \frac{dy}{dt}\, dt$. The line integral with respect to z is: $\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t)) \frac{dz}{dt}\, dt$d

Parametrization Independence

Line Integrals w.r.t. Arc Length Along C (Scalar Function) Line integrals of a scalar function with respect to arc length along C is independent of the **parametrization** of C and the **orientation** of C.

- **Independence of Parametrization**: If a line integral is rewritten as $\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{r}'(t)\|\, dt$, each element is geometrically the same, whether it be the values of f at (x, y, z) or the different representation of arc length geometrically.
- **Independence of Orientation of C**: A curve C can be traversed in 2 ways. However, again, sub-arc lengths are still the same and again f is the same.

Line Integrals w.r.t. x/y/z Along C (Scalar Function) Line integrals of a scalar function with respect to x/y/z along C is independent of the **parametrization** of C.

- **Independence of Parametrization**: Again, parametrization does not affect the geometries, so it is evaluated the same.
- **Influenced by Orientation of C**: A reversal of traversal path is denoted as $-C$. Due to this reversal, path elements in the x direction becomes negative, changing the sign of the reversed traversal line integral: $\int_{-C} f(x, y, z)\, dx = -\int_C f(x, y, z)\, dx$.

Line Integral of a Vector Field

Definition and Properties

Given a vector field $\vec{F}(x, y, z)$ and a smooth curve C parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$, the line integral of $\vec{F}$ along C is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \langle F_x, F_y, F_z \rangle \cdot \langle dx, dy, dz \rangle = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

- The expression $\int_C \vec{F} \cdot d\vec{r}$ represents the integral of the tangential component of $\vec{F}$ along the curve C. F_x and the rest of the components are parametrized in terms of t.
- Since F_x and the rest of the components are parametrized in terms of t, $\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$, so then for each component's differential, $\frac{dx}{dt} dt = dx$.
- This line integral represents **work**.

Properties:

- *Linearity*: $\int_C (a\vec{F} + b\vec{G}) \cdot d\vec{r} = a \int_C \vec{F} \cdot d\vec{r} + b \int_C \vec{G} \cdot d\vec{r}$, where a and b are constants.
- *Path Reversal*: If $-C$ is the curve C traversed in the opposite direction, then $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$ (**influenced by orientation**).
- *Path Additivity*: If C is composed of two curves C_1 and C_2 joined end-to-end, then $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$.

Fundamental Theorem of Line Integrals

The following Fundamental Theorem of Line Integrals applies only to **conservative vector fields**.

If $\vec{F}$ is a conservative vector field, meaning $\vec{F} = \nabla\phi$ for some scalar function ϕ (called the potential function), and C is a smooth curve from point (x_0, y_0) to point (x_1, y_1), then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{(x_0,y_0)}^{(x_1,y_1)} \nabla\phi \cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

- This theorem states that the line integral of a gradient field depends only on the endpoints of the curve and not on the path taken.
- If the curve C is closed, that is $(x_1, y_1) = (x_0, y_0)$, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for any gradient field $\vec{F}$.

The following statements are equivalent (either all true or all false):

Theorem. If $\mathbf{F}$ is a continuous vector field on some open, connected region D in 2-space, then the following statements are equivalent:

(i) $\mathbf{F}$ is conservative on D.

(ii) For any piecewise smooth, closed curve C lying in D,

$$\int_C \mathbf{F} \bullet d\mathbf{r} = 0.$$

(iii) Work integrals of $\mathbf{F}$ along piecewise smooth curves in D are independent of path; their values depend only on the endpoints of the curve and not on the curve itself.

- **Note**: Beware of overgeneralization on iii. Just because the line integral over two different curves that have the same beginnings and endings yield equal values doesn't mean that $\vec{F}$ is conservative.

Conservative Vector Fields

General strategy for finding a potential ϕ for the conservative field $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$

Must solve the pair of partial differential equations:

$$\phi_x = f \text{ and } \phi_y = g.$$

Method 1:

(i) Integrate f with respect to x, to obtain

$$\phi(x,y) = \int f(x,y)dx + K(y),$$

where K is an arbitrary function of y.

(ii) Plug this expression for ϕ into the equation $\phi_y = g$ to determine what K must be.

Method 2:

(i) Integrate f with respect to y, to obtain

$$\phi(x,y) = \int f(x,y)dy + K(x),$$

where K is an arbitrary function of x.

(ii) Plug this expression for ϕ into the equation $\phi_x = f$ to determine what K must be.

Exercise. Think about how these methods might fail in the cases where the original vector field is *not* conservative.

Definition and Properties

A vector field $\vec{F}$ is said to be **conservative** if it is the **gradient of some scalar function**, called a **potential function**.

- That is, $\vec{F} = \nabla f$ for some scalar function f. More often, f is denoted as a **potential function** $\phi(x, y, z)$.

Key Properties:

- **Path Independence**: The line integral of $\vec{F}$ between two points is independent of the path taken. The line integral depends only on the endpoints.
  - The **Fundamental Theorem of Line Integrals** may be used to evaluate such path-independent line integrals.
  - The direction of path traversal still matters.
- **Closed Line Integral is Zero**: The integral of $\vec{F}$ over any closed loop/path is always zero. Notably, this is not necessarily a circle, but instead that the line integral around every possible closed path in that region must be zero, regardless of the shape.

$$\oint \vec{F} \cdot d\vec{r} = 0$$

- **The curl of $\vec{F}$ is zero everywhere in a simplyconnected domain**: $\nabla \times \vec{F} = 0$.
  - This is an **if and only if** condition. Therefore, similarly, if $\nabla \times \vec{F}$ and the field is defined on a simply connected domain, then $\vec{F}$ will definitely be a conservative vector field.

Simple Curves and Simply Connected Regions

A **simple curve** is a continuous curve in space that does not intersect itself, except possibly at its endpoints.

A region D is **simply connected** if it is "hole-free". That is, every closed curve within D can be continuously deformed into another curve, **down to a point**, while remaining entirely within D.

- Thus, a shape like an annulus is **not simply connected**.

Finding the Potential Function of a Conservative Vector Field

A vector field $\mathbf{F}$ is said to be **conservative** if there exists a scalar potential function ϕ such that $\mathbf{F} = \nabla\phi(x, y, z)$.

Integration of Partial Derivatives Let $\mathbf{F}(x, y) = \langle F_x, F_y \rangle$. Then if $\mathbf{F} = \nabla\phi$, by definition, $\frac{\partial\phi}{\partial x} = F_x$ and $\frac{\partial\phi}{\partial y} = F_y$.

1. **Integrate $F_x(x, y)$ with respect to x:** $\phi(x, y) = \int F_x(x, y)\, dx + g(y)$, where $g(y)$ is an unknown function of y.
2. **Differentiate ϕ with respect to y and set it equal to $F_y(x, y)$:** $\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(\int F_x(x, y)\, dx + g(y)\right) = F_y(x, y)$.
3. **Solve for $g(y)$ from this equation.**
4. **Combine the results** to obtain the full potential function $\phi(x, y)$. (An arbitrary constant C may appear, which does not affect the gradient. Any convenient value may be

plugged in for C.)

Conservative Field Test Theorem

One key property of **conversative vector fields** is that its **curl** is zero in a simply-connected domain. This is an "if and only if" statement, meaning both directions of the implication holds true. Therefore, **a vector field having zero curl in a simply-connected region** is sufficient to show that the vector field is conservative (at least in that region).

- Succintly: If a continuously differentiable vector field $\mathbf{F}$ on an open, simply connected region D satisfies $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is conservative on D. Conversely, if $\mathbf{F}$ is conservative on D, then $\nabla \times \mathbf{F} = 0$ in D.

In 3D, the curl of a vector field $\mathbf{F} = \langle F_x, F_y, F_z \rangle$ is a vector field defined as $\nabla \times \mathbf{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle = \vec{0}$. For a 2D vector field (a vector field over an xy plane), $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$, the curl simplifies to $\text{curl}\, \mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ for a conservative field over a simply-connected region.

Note: It is well known that all inverse-square fields in the form $\mathbf{F}(\mathbf{r}) = k\frac{\mathbf{r}}{|\mathbf{r}|^3}$ are conservative.

Closed Curves and Closed Line Integrals

Definition and Properties

A **closed line integral** (also called a contour integral) is an integral taken over a closed curve. Mathematically, if **C** is a closed curve and **F** is a vector field, the closed line integral is written as $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = F_x(x,y)\hat{\imath} + F_y(x,y)\hat{\jmath}$ is a vector field.

This integral measures how much the vector field flows along the path **C**. If **C** encloses some area, the behavior of the vector field inside can often be analyzed using **Green's Theorem**.

Note: The only conditions are that the region D must be simply connected, closed, and oriented positively + the vector field has continuous partial derivatives for each of its derivatives within/on the region. The vector field **does not need to be conservative.**

- In fact, Green's Theorem is useful in cases when the curl is nonzero; if the vector field is conservative, the curl becomes zero.

Evaluating Closed Line Integrals Without Green's Theorem

1. **Parametrize the Curve** C: Write a parametrization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [a, b]$ such that it describes one full traversal of C.
2. **Compute the Integrand**: Substitue $x(t)$ and $y(t)$ into $\mathbf{F}$, and then calculate $\mathbf{F}(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle$.
3. **Integrate from** $t = a$ **to** $t = b$: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b [\mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)]\, dt$.

Green's Theorem for Simply Connected Regions

Green's Theorem relates a line integral around a simple closed curve C to a double integral over the plane region D enclosed by C. Intuitively, it equates the **circulation** along the boundary of a region (line integral) with the **cumulative rotation/curl** within that region (double integral of the curl).

Theorem Statement: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let D be the region bounded by C. A vector field $\vec{F}(x, y)$ in two dimensions can be written as $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ and $d\mathbf{r} = \langle dx, dy \rangle$. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint_C (P\, dx + Q\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- The notation $\oint_C$ denotes that the curve C is closed and positively oriented (counterclockwise).
- When it is a conservative vector field, this form is literally the closed line integral of a total differential.
- The curl is the $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ integrand.

Corollaries Corollary 1: Computing the area of a planar region whose boundary is known (particularly when the boundary C is easier to parametrize than describing the region R itself).

The right side of Green's Theorem becomes equivalent to Area(R) when $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Working backwards, an easy result is $P(x, y) = -\frac{y}{2}$ and $Q(x, y) = \frac{x}{2}$, so that $\frac{\partial Q}{\partial x} = \frac{1}{2}$ and $\frac{\partial P}{\partial y} = -\frac{1}{2}$.

$$\text{Area}(R) \;=\; \frac{1}{2} \oint_C (x\, dy \;-\; y\, dx)$$

Corollary 2: Green's Theorem also shows that if a vector field has zero curl in a simply connected region, its line integral around any closed curve in that region is also zero. (This is a property of conservative vector fields, but Green's Theorem also shows this mathematically.)

In other words, if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, the integrand of the right side of Green's Theorem is zero and therefore the closed line integral $\oint_C (P\,dx + Q\,dy) = 0$.

A Generalized Green's Theorem for Multiply Connected Regions

Green's Theorem, in its basic form, relates a line integral around a simple closed curve C to a double integral over the region R that C encloses. The generalized form extends this to regions with "holes," or more precisely, regions that are not simply connected.

Theorem Statement: If R is a region in the plane with a boundary that consists of a finite number of simple closed curves, and if P and Q have continuous first-order partial derivatives on an open region that contains R, then the generalized form of Green's Theorem can be stated as:

$$\oint_{C_0} (P\,dx + Q\,dy) - \sum_{i=1}^{n} \oint_{C_i} (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where

- C_0 is the outer boundary of R, oriented counterclockwise.
- $C_1, C_2, \ldots, C_n$ are the inner boundaries (the "holes") of R, oriented clockwise.

Intuitively, when you consider a region with holes, the line integrals along the inner boundaries are subtracted because their orientations are opposite to that of the outer boundary. This can be thought of as the circulation around the holes "canceling out" part of the circulation induced by the outer boundary.

Note 1: For Green's Theorem to by applied to multiply-connected regions, the outer curve must be oriented **counterclockwise** (positively) while the inner curves surrounding the holes must be oriented **clockwise** (negative). **Note 2 (Shortcut)**: If the integrand is a constant value, then the line integral over the closed loop will be that constant multiplied with the area of the region (after subtracting the holes).

Stokes' Theorem

Stokes' Theorem generalizes Green's Theorem to three dimensions

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

curl becomes a vector quantity in three dimensions

3. Surface Integrals

Scalar Surface Integrals

Definition and Properties

The **surface integral** takes summing up values over a flat region (e.g., "heights" via functions over the xy-plane) and **extends it to curved surfaces in 3-space**. Now, instead, we are dividng up a surface into tiny patches and assigning each patch in space a **scalar weight**, which is a function $f(x, y, z)$ multiplied with the patch's area.

Essentially, it is a double integral with bounds determined by the parametrized surface (e.g., in terms of u, v), with an area transformation element dS. The key to solving surface integrals is to collapse a 3D problem into a 2D one. The local surface area scaling factor occurs as an infinitesimal rectangular area $dudv$ stretches when mapped onto a curved surface in $\mathbb{R}^3$.

Let σ be a smooth parametric surface parametrized by the vector equation $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where (u, v) varies over some region R in the uv-plane. If $f(x, y, z)$ is a continuous function on σ, then the **surface integral** is:

$$\iint_\sigma f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v)) ||\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)||\, dA$$

- $||\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)||$ is the Jacobian determinant for the local area transformation. That is, $dS = ||\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|| dudv$.
- For the special case where $f(x, y, z) = f(x, y, g(x, y))$, the transformation factor becomes $dS = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA$.

Relation to Surface Area The surface integral is an extension of the **double integral to find surface area**. That is, it is a more generalized form with function $f(x, y, z)$; the double integral to find surface area is simply the **surface integral with** $f(x, y, z) = 1$.

$$S = \iint_D ||\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|| dA$$

Choosing Parametrization & Problem-Solving Framework

When given a surface like a plane, it's easiest to find the intercepts and see the surface as the triangle in 3-space connecting those three points.

There are two common approaches:

1. The surface is given as a graph, like for example $z = g(x, y)$. Then, $dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$. You are essentially collapsing the surface into the xy-plane and can rewrite the integrand as $f(x, y, z) = f(x, y, g(x, y))$.
2. Use (u, v) to parametrize both $f(x, y, z)$ and the surface σ. After parametrizing the surface, figure out the bounds as well as the area element dS.

Problem-Solving Procedure:

1. Choose an appropriate parametrization. Common options include using polar or using 2D coordinates (e.g., x & y). This will determine whether you use approach 1 or 2.
2. Now determine the **three key pieces** of information: the bounds, the area scale factor dS, and the parametrized weight $f(x(u, v), y(u, v), z(u, v))$.
   1. **Bounds**: This will depend directly on the surface σ. For example, if the surface is boxed within the bounds $x = a$, $x = b$, $z = c$, $z = d$, then it may be easiest to use these given bounds and project the surface onto the xz-plane (adjusting the equation of the surface to be $y(x, z)$). At other times, you'll need to pick u, v such that $\mathbf{r}(u, v)$ covers every single point on surface σ.
   2. **Scale Factor**: In general, $dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$, with the special case of $dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$.
   3. **Integrand/Weight**: Substitute u/v in for $f(x, y, z) \to f(x(u, v),\ y(u, v),\ z(u, v))$.
3. Evaluate the double integral.

Vector Surface Integrals (Flux Across Surfaces)

Flux is a concept in quantifies the "flow" of a vector field across a surface. For example, it measures the net rate at which the quantity represented by the vector field (e.g., fluid, electric field lines) passes through the surface.

Definition and Properties

Intuitively, imagine a vector field F representing the velocity of a fluid in space. If we place a surface S (like a net) within this fluid, the *flux* of F across S represents the net volume of fluid passing through the surface per unit time.

- If the vector field F is largely perpendicular (normal) to the surface, the flow *through* the surface is high.
- If the vector field F is largely parallel (tangent) to the surface, the flow *through* the surface is low.
- The direction matters: flow in one direction across the surface can cancel out flow in the opposite direction.

The **formal definition is as follows**: The flux of a vector field $\mathbf{F}(x, y, z)$ across an oriented surface σ with unit normal vector field $\mathbf{N}(x, y, z)$ is defined by the surface integral:

$$\Phi = \iint_\sigma \mathbf{F}(x, y, z) \cdot \mathbf{N}(x, y, z) \, dS$$

- If the orientation is reversed, $\mathbf{N}(x, y, z)$ changes sign, and so flux **also** changes sign.
- At times, it will be easier to compute $\mathbf{N}(x, y, z)$ by finding the gradient and dividing by its magnitude.

Role of Surface Orientation

To define flux, the surface σ must be *orientable*. An orientable surface is one for which we can consistently define a continuous unit normal vector field $\mathbf{N}$ at every point $p \in \sigma$.

- This choice of $\mathbf{N}$ defines an *orientation* for the surface, effectively distinguishing its two sides (e.g., "inside" vs. "outside", or "up" vs. "down").
  - For non-orientable surfaces like the Möbius strip, there is no consistent way to define a "through" direction globally, making the standard definition of flux problematic.

There are **two scenarios**:

1. The surface is not parametrized $\longrightarrow$ orientation is determined by the sign of its gradient.
2. The surface is parametrized $\longrightarrow$ orientation is determined by $\mathbf{r}_u \times \mathbf{r}_v$.
   - For the special case of collapsing onto the xy-plane or some other equivalent (when $z = g(x, y)$), $\mathbf{n} \, dS$ becomes $\langle -z_x, -z_y, 1 \rangle$ for upward-facing normals.

Computation via Parametrization

If the surface σ is parametrized by a function $\mathbf{r}(u, v)$ for (u, v) in a domain D, the calculation proceeds as follows:

- Compute the tangent vectors: $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$.
- Compute the normal vector to the surface (not necessarily unit length): $\mathbf{r}_u \times \mathbf{r}_v$.
  - **Note**: This is because $\mathbf{N}(x(u, v), y(u, v), z(u, v)) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ and $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$. Multiplying together cancels the $\|\mathbf{r}_u \times \mathbf{r}_v\|$, becoming $\mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)$.
- The flux integral is:

$$\Phi = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

Note on Signs: The orientation of the unit normal is important. If the problem specifies one or the other, ensure that the sign on $\hat{k}$ is correct.

Flux and Closed Surfaces (Divergence/Gauss's Theorem)

Theorem Statement: Let E be a simple solid region in $\mathbb{R}^3$ whose boundary surface S is piecewise smooth and has a positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous first-order partial derivatives on an open region containing E. Then:

$$\Phi = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E (\nabla \cdot \mathbf{F}) \, dV$$

Conditions for Applicability

- The vector field $\mathbf{F}$ must have continuous first-order partial derivatives in a region containing E. This ensures that the divergence $\nabla \cdot \mathbf{F}$ is well-defined and integrable.
- The region E must be a *simple solid region*, often referred to as a regular region or an elementary region. This generally means it's a bounded solid that can be described simultaneously in Cartesian, cylindrical, or spherical coordinates without breaking it into too many pieces.
- The boundary surface S must be *piecewise smooth* and *closed*. Closed means it completely encloses the volume E. Piecewise smooth means it consists of a finite number of smooth surfaces joined together.
- The orientation of the surface S must be the *positive (outward)* orientation. The normal vectors $\mathbf{n}$ must point away from the enclosed volume E.

Divergence and Flux Density

The **net outward flux density** is the *flux per unit volume at a point*. This definition gives the divergence of a vector field $\mathbf{F}$ physical significance/meaning.

Consider a small sphere G of volume $\mathrm{vol}(G)$ centered at a point P_0.

The **total outward flux** $\Phi(G)$ through the surface $\sigma(G)$ of the sphere is:

$$\Phi(G) = \iint_{\sigma(G)} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_G (\nabla \cdot \mathbf{F}) \, dV$$

If $\nabla \cdot \mathbf{F}$ is approximately constant near P_0, then:

$$\Phi(G) \approx \big(\nabla \cdot \mathbf{F}(P_0)\big) \, \mathrm{vol}(G)$$

Rearranging gives:

$$\nabla \cdot \mathbf{F}(P_0) = \lim_{\text{vol}(G) \to 0} \frac{\Phi(G)}{\text{vol}(G)}$$

Gauss's Law for Inverse-Square Fields

Stokes' Theorem

Stokes' Theorem applies to an *open surface with a closed boundary loop*. For example, imagine a paraboloid: it is an open surface and it has a circle/ellipse at one end.

The theorem links the **curl of a vector field over an oriented surface** S with the **line integral of the vector field over boundary curve** C **of the surface.**

Theorem Statement: Let S be an oriented, piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation relative to the surface's orientation. Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives on an open region in $\mathbb{R}^3$ containing S. Then, Stokes' Theorem states:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

Components and Conditions

- **Vector Field** $\mathbf{F}$: This is a vector field, typically denoted as $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. For Stokes' Theorem to apply, the component functions P, Q, R must have continuous first-order partial derivatives in a region containing the surface S.
- **Surface** S: This is an oriented, piecewise-smooth surface in $\mathbb{R}^3$.
  - *Oriented* means that the surface has a chosen normal vector $\mathbf{n}$ at each point, varying continuously across the surface. This defines a "top" and "bottom" or "inside" and "outside".
  - *Piecewise-smooth* means the surface can be decomposed into a finite number of smooth pieces.
- **Boundary Curve** C: This is the simple, closed, piecewise-smooth curve that forms the edge or boundary of the surface S.
  - *Simple* means the curve does not intersect itself.
  - *Closed* means the curve starts and ends at the same point.
- **Positive Orientation of** C: The orientation of the boundary curve C is linked to the orientation of the surface S (defined by the normal vector $\mathbf{n}$. The positive orientation is determined by the right-hand rule: if you curl the fingers of your right hand in the direction of the curve C, your thumb must point in the direction of the surface normal

$\mathbf{n}$. Traversing C in the positive direction keeps the surface S on your left, relative to the direction of $\mathbf{n}$.

- **Curl of $\mathbf{F}$**: The curl, denoted $\nabla \times \mathbf{F}$, measures the infinitesimal rotation or circulation of the vector field at a point. It is defined as:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Additional Notes

- Stokes' Theorem is a generalization of Green's Theorem to three dimensions. Green's Theorem is a special instance where only the $\mathbf{k}$ component is nonzero.
- The **orientation** of the *curve* must **align** with the orientation of the *surface's normals*. This is done via the right hand rule applied to the curve to determine the parametrization in a manner ensuring orientation alignment.
- The right side of Stokes' Theorem, $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, is equal for any two surfaces σ_1 and σ_2 as long as both surfaces **share the same positively oriented boundary** (equality of the left side of Stokes' Theorem).

Curl and Circulation Density

The **circulation of a vector field $\mathbf{F}$** around a simple closed curve C_a is defined by the line integral $\oint_{C_a} \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector to the curve C_a. This integral measures the tendency of the vector field $\mathbf{F}$ to "circulate" around the curve C_a.

The circulation density of $\mathbf{F}$ around C_a is defined as the circulation per unit area:

$$\frac{\oint_{C_a} \mathbf{F} \cdot \mathbf{T}\, ds}{\text{Area}(\sigma_a)}$$

- Here, σ_a represents a small surface bounded by the curve C_a. The circulation density gives a measure of the intensity of circulation in the vicinity of the surface σ_a.

The curl of a vector field at a point can be interpreted as a **measure of the circulation density through an axis at that point**. If $\text{curl}\, \mathbf{F} \cdot \mathbf{n}$ is continuous at a point P, we can consider a small surface σ_a containing P with boundary curve C_a and unit normal vector $\mathbf{n}$. As the area of this surface shrinks to zero ($a \to 0$), the component of the curl in the direction $\mathbf{n}$ at point P is given by the limit of the average value of $(\text{curl}\, \mathbf{F}) \cdot \mathbf{n}$ over σ_a:

$$\text{curl}\, \mathbf{F}(P) \cdot \mathbf{n} = \lim_{a \to 0} \left(\frac{1}{\text{area}(\sigma_a)} \iint_{\sigma_a} (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, dS \right)$$

Using Stokes' Theorem, we can replace the surface integral with the line integral for circulation:

$$\text{curl } \mathbf{F}(P) \cdot \mathbf{n} = \lim_{a \to 0} \frac{\oint_{C_a} \mathbf{F} \cdot \mathbf{T} \, ds}{\text{area}(\sigma_a)}$$

This relationship demonstrates that $\text{curl } \mathbf{F} \cdot \mathbf{n}$ represents the *circulation density* of the vector field $\mathbf{F}$ at point P in the direction of the unit normal vector $\mathbf{n}$. It quantifies the "swirl" or rotational tendency of the field around an axis defined by $\mathbf{n}$ at point P.

Additional Notes:

- The magnitude of the curl vector, $|\text{curl } \mathbf{F}|$, represents the maximum possible circulation density at that point. This occurs when $\mathbf{n}$ and $\text{curl } \mathbf{F}$ are aligned (due to dot product).
- The curl is also the *axis of rotation* for circulation at each point.

Summary

Computation of Surface Integrals

When computing surface integrals, there are only a few scenarios where the "$\mathbf{n} \, dS$" notation should be kept. Either **1)** the surface was expressed explicitly in the form $z = g(x, y)$ (or some equivalent form with x/y), in which the entire term becomes $\langle -z_x, -z_y, 1 \rangle$ if the orientation of the normal is upward ($\langle z_x, z_y, -1 \rangle$ if normal is pointed downward) or **2)** the surface is in some convenient form like a sphere where unit normals happen to be the coordinate divided by distance.

Symmetry / odd functions can become very convenient around circular regions when collapsed to 2D after parametrization. For example, the double integral of x or y around the circular region goes to zero. **Notably**, this doesn't apply when facing x^2 or y^2.

In all other situations, parametrization should make the computation easier.

Stokes' Theorem

Stokes' Theorem is essentially the same thing as Green's Theorem, just with all components of curl. The most important thing is that **it should only be applied when the orientation of the surface & closed boundary curve point in the *same* direction**.

To determine the orientation of the closed boundary curve, use the right hand rule. Clockwise points down, counterclockwise points up.

4. Vector Calculus Summary

Line Integral

- work - open line integral, may use *fundamental theorem of line integrals* if conservative field
- circulation - closed line integral
  - Green's Theorem / Stokes' Theorem
- surface area - integrate height function along a curve
- mass of string - integrate linear density along curve

Surface Integral

- surface area - integrate with integrand of 1, applying the proper transformation to get a double integral
- mass of a surface - integrate scalar density weight across surface
- flux - integrate the normal components of a vector field to a surface; positive means outward flow / source, negative means inward flow / sink
- summing circulation for closed line integral / work - integrate the curl of a vector field dotted with the normal of the surface; positive means counterclockwise circulation, negative means clockwise circulation

Divergence of F is flux density → the sum of flux density across a volume gives the flux through the surface

Curl of F is circulation density → the sum of circulation density across a surface gives the work done by the force across closed boundary of surface

- Curl gives the axis of rotation for circulation
- Magnitude of curl is the maximum strength of rotation (when it's aligned with unit vector **n**)

Theorems

- Fundamental Theorem of Line Integrals
- Divergence Theorem
- Stokes' Theorem